UPSC Civil Services Main 1979 - Mathematics Algebra

Brij Bhooshan

Asst. Professor

B.S.A. College of Engg & Technology

Mathura

Question 1(a) *Define the centre of a group and show that a group of prime power has a non-trivial centre. Hence or otherwise prove that a group of order p^2 is abelian for every prime p.*

Solution. See theorem 2.11.2 Page 86 of Algebra by Herstein. See also the corollary to the theorem. ■

Question 1(b) *Show that any two cyclic groups of the same order are isomorphic.*

Solution. Let G_1 and G_2 be two cyclic groups, $G_1 = \langle a \rangle, G_2 = \langle b \rangle$.

Case 1. G_1 and G_2 are of infinite order. Define $T : G_1 \longrightarrow G_2$ by $T(a^r) = b^r$.

1. T is a homomorphism. Let $\alpha, \beta \in G_1$, then $\alpha = a^r, \beta = a^s$ for some $r, s \in \mathbb{Z}$. $T(\alpha\beta) = T(a^{r+s}) = b^{r+s} = b^r b^s = T(\alpha)T(\beta)$.
2. T is one-one. $\alpha \neq \beta \Rightarrow r \neq s \Rightarrow T(\alpha) \neq T(\beta)$.
3. T is onto is obvious.

Thus $G_1 \simeq G_2$.

Case 2. ord(G_1) = ord(G_2) = n. Again define $T(a^r) = b^r$. Let $\alpha, \beta \in G_1 \Rightarrow \alpha = a^r, \beta = a^s \Rightarrow T(\alpha\beta) = T(a^{r+s})$. If $r + s \equiv t \mod n$, then $T(\alpha\beta) = T(a^t) = b^t = b^{r+s} = T(\alpha)T(\beta)$, so T is a homomorphism. If $\alpha \neq \beta$, then $r \not\equiv s \mod n$, therefore $T(\alpha) \neq T(\beta)$. T is onto is obvious, so $G_1 \simeq G_2$. ■

Question 1(c) *If* $\mathbb{Q}$ *is the field of rational numbers, prove that* $\mathbb{Q}(\sqrt{2},\sqrt{3}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$, *where* $\mathbb{Q}(a_1, a_2)$ *is the smallest subfield of the real field containing* $\mathbb{Q}, a_1, a_2$.

Solution. Clearly $\mathbb{Q}(\sqrt{2},\sqrt{3}) \supseteq \mathbb{Q}(\sqrt{2}+\sqrt{3})$. We shall prove that the degree of $\mathbb{Q}(\sqrt{2},\sqrt{3})$ over $\mathbb{Q}$ = degree of $\mathbb{Q}(\sqrt{2}+\sqrt{3}) = 4 \Rightarrow \mathbb{Q}(\sqrt{2},\sqrt{3}) = \mathbb{Q}(\sqrt{2}+\sqrt{3})$. Let $x = \sqrt{2}+\sqrt{3}$ so that $x - \sqrt{2} = \sqrt{3}$. Squaring both sides, we get $x^2 - 2\sqrt{2}x + 2 = 3 \Rightarrow x^2 - 1 = 2\sqrt{2}x \Rightarrow x^4 - 2x^2 + 1 = 8x^2 \Rightarrow x^4 - 10x^2 + 1 = 0$. Now $x^4 - 10x^2 + 1$ is irreducible over $\mathbb{Q}$ — it has no rational roots, as ± 1 are not roots of the polynomial. Another way to see this is to observe that $t^2 - 10t + 1$ has no rational roots, so is irreducible. Thus $(\mathbb{Q}(\sqrt{2}+\sqrt{3}) : \mathbb{Q}) = 4$.

Now $\sqrt{3} \notin \mathbb{Q}(\sqrt{2})$, because if $\sqrt{3} = a + b\sqrt{2}, a, b \in \mathbb{Q}$, then $3 = a^2 + 2\sqrt{2}ab + 2b^2$, which would imply that $\sqrt{2} \in \mathbb{Q}$. Thus $(\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}(\sqrt{2})) = 2 \Rightarrow (\mathbb{Q}(\sqrt{2},\sqrt{3}) : \mathbb{Q}) = 4$. ∎