UPSC Civil Services Main 1980 - Mathematics Algebra

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Mathura

Question 1(a) *Let R be a relation on a non-empty set A and for every $y \in A$ let*

$$B_y = \{x \mid (x, y) \in R\}$$

Show that the collection $\{B_y \mid y \in A\}$ is a partition if and only if R is an equivalence relation.

Solution. Let R be an equivalence relation. We need to show that $\{B_y \mid y \in A\}$ is a partition of A. Let $a, b \in A$. It is enough to show that if $x \in B_a \cap B_b$, then $B_a = B_b$. Let $y \in B_a$, then $(y, a) \in R$. As $x \in B_a, (x, a) \in R$, and $x \in B_b, (x, b) \in R$. Now by symmetry and transitivity of R, $(x, a) \in R \Rightarrow (a, x) \in R$, and $(y, a) \in R, (a, x) \in R \Rightarrow (y, x) \in R$, and now $(x, b) \in R \Rightarrow (y, b) \in R \Rightarrow y \in B_b$ so $B_a \subseteq B_b$. Similarly $B_b \subseteq B_a \Rightarrow B_a = B_b$. So $B_a \cap B_b = \emptyset$ or $B_a = B_b$ for all $a, b \in A$. Since $A = \bigcup_{a \in A} B_a$, the set $\{B_y \mid y \in A\}$ is a partition of A.

The converse is **false**. Let $A = \{a, b\}$ and let $R = \{(a, b), (b, a)\}$. Then $B_a = \{b\}, B_b = \{a\}$, which is a partition of A, but R is not an equivalence relation, as it is not reflexive or transitive. ∎

Question 1(b) *Let I be the set of integers and let R be a relation on $I \times I$ defined by $(m, n)R(p, q)$ if and and only if $mq = np$. Show that R is an equivalence relation and identify the partition generated by R.*

Solution. The question as stated is not correct. $(2, 3)R(0, 0)$ and $(0, 0)R(1, 2)$, but $(2, 3)R(1, 2)$ is false, as $2 \times 2 \neq 3 \times 1$. However if we define $I^* = I - \{0\}$, and R is defined as a relation over $I \times I^*$ with the above definition, then we can show that it is an equivalence relation.

Reflexive: Clearly $(m, n)R(m, n)$, as $mn = nm$. **Symmetric**: $(m, n)R(p, q) \Rightarrow (p, q)R(m, n)$ because $mq = np \Rightarrow qm = pn$. **Transitive**: If $(m, n)R(p, q)$, then $mq = np$. If $(p, q)R(r, s)$ then $ps = qr$. Thus $mqs = nps = nqr \Rightarrow ms = nr$ because $q \neq 0$. Thus $(m, n)R(r, s)$.

Clearly $B_{(m,n)} = \{(a, b) \mid mb/n \in I, a = mb/n\}$. In fact this is isomorphic to the set of all rational numbers. ∎

Question 1(c) *Let A and B be any two non-empty sets. Show that the collections of all mappings from A to B is a proper subset of the collection of all relations from A to B.*

Solution. Let f be a function from A to B. Then the set $\{(a, f(a)) \mid a \in A\}$ is a relation from A to B. Thus the set of all functions is a subset of the set of all relations. It is a proper subset if B has more than one element — the $R = A \times B$ is a relation from A to B, but is not a function. ∎

Question 2(a) *Show that the set of all transformations $x \longrightarrow \frac{ax+b}{cx+d}, a, b, c, d \in \mathbb{R}, ad-bc = 1$ is a group. Examine whether the restricted set of transformations where a, b, c, d are integers subject to the same constraint will form a group.*

Solution. Let G be the set of all transformations, and H be the set of all transformations where a, b, c, d are integers.

- G is nonempty. $I : x \longrightarrow x$ is in G, with $a = d = 1, b = c = 0$.
- If $T_1(x) = \frac{a_1x+b_1}{c_1x+d_1}, T_2(x) = \frac{a_2x+b_2}{c_2x+d_2}$, then $(T_2 \circ T_1)(x) = \frac{a_2\frac{a_1x+b_1}{c_1x+d_1}+b_2}{c_2\frac{a_1x+b_1}{c_1x+d_1}+d_2} = \frac{(a_1a_2+b_2c_1)x+a_2b_1+b_2d_1}{(c_2a_1+d_2c_1)x+c_2b_1+d_1d_2}$ is an element of G because $(a_1a_2 + b_2c_1)(c_2b_1 + d_1d_2) - (a_2b_1 + b_2d_1)(c_2a_1 + d_2c_1) = (a_1d_1 - b_1c_1)(a_2d_2 - b_2c_2) = 1$.
- $T \circ I = I \circ T = T$ for every $T \in G$.
- Given $T(x) = \frac{ax+b}{cx+d}, T^{-1}(x) = \frac{dx-b}{-cx+a}$ is the inverse of T. In fact $T \circ T^{-1} = I = T^{-1} \circ T$, this can be checked by substituting in the above formula and setting $ad - bc = 1$.
- The operation is clearly associative.

Thus G is a group. H is clearly closed, and $T \in H \Rightarrow T^{-1} \in H$, so H is a group.

In fact G is isomorphic to $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, a, b, c, d \in \mathbb{R}\}$ and H is isomorphic to $\{\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1, a, b, c, d \in \mathbb{Z}\}$. ∎

Question 2(b) *Show that a group of order 15 is cyclic.*

Solution. By Cauchy's theorem, a group of order 15 has an element a such that the order of $a = o(a) = 5$ and G has an element b such that $o(b) = 3$. Let $H = \langle a \rangle$, the group generated by a, and $K = \langle b \rangle$, the group generated by b. Then $H \cap K = \{e\}$, because $x \in H \Rightarrow x = e$ or $o(x) = 5$, and $x \in K \Rightarrow x = e$ or $o(x) = 3$.

Since G has a unique subgroup of order 5, and a unique subgroup of order 3, by Sylow's theorems, H, K are normal subgroups of G. Then $ab = ba$ because $aba^{-1} \in H, ba^{-1}b^{-1} \in K \Rightarrow aba^{-1}b^{-1} \in H \cap K \Rightarrow ab = ba$. Hence $o(ab) = 15 \Rightarrow G$ is cyclic.

Here we have used the fact that if in a group G, $o(x) = l, o(y) = m, (l, m) = 1, xy = yx \Rightarrow o(xy) = lm$. ∎

Question 2(c) *Let G be a finite group and $F = \{f : G \longrightarrow \mathbb{C}\}$ be the set of all complex valued functions on G. If addition and multiplication in F are defined for $f, g \in F$ by*

$$\begin{aligned} (f+g)(x) &= f(x) + g(x) \\ (fg)(x) &= \sum_{y \in G} f(xy^{-1})g(y) \end{aligned}$$

for every $x \in G$, show that F is a ring with identity.

Solution. F is a commutative group for addition is obvious, as the 0 function is the additive identity, $-f$ is the inverse of f, and $+$ is commutative and associative in $\mathbb{C}$.

Let $e^* : F \longrightarrow \mathbb{C}$ be defined by $e^*(x) = 1$ if $x = e$, 0 otherwise. Then

$$(f.e^*)(x) = \sum_{y \in G} f(xy^{-1})e^*(y) = f(xe^{-1}) = f(x)$$

and

$$(e^*.f)(x) = \sum_{y \in G} e^*(xy^{-1})f(y) = f(x)$$

because $e^*(xy^{-1}) = 1 \Leftrightarrow xy^{-1} = e \Leftrightarrow x = y$, and all other terms vanish.

1. Thus e^* is the multiplicative identity of F.
2. $f(g+h) = \sum_{y \in G} f(xy^{-1})(g+h)(y) = fg + fh$
3. $(f+g)h = fh + gh$

Thus F is a ring with identity element. ∎