UPSC Civil Services Main 1981 - Mathematics Algebra

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Question 1(a) *Let $f : X \longrightarrow Y$ be a bijective mapping and let $A \subseteq X, B \subseteq X$, show that $f(A \cap B) \subseteq f(A) \cap f(B)$ and $f(A \cup B) = f(A) \cup f(B)$. Examine whether $f^{-1}(A \cap B)$ is properly contained in $f^{-1}(A) \cap f^{-1}(B)$.*

Solution.

$$\begin{aligned} A \cap B \subseteq A &\Rightarrow f(A \cap B) \subseteq f(A) \\ A \cap B \subseteq B &\Rightarrow f(A \cap B) \subseteq f(B) \\ &\Rightarrow f(A \cap B) \subseteq f(A) \cap f(B) \end{aligned}$$

$$\begin{aligned} A \subseteq A \cup B &\Rightarrow f(A) \subseteq f(A \cup B) \\ B \subseteq A \cup B &\Rightarrow f(B) \subseteq f(A \cup B) \\ &\Rightarrow f(A) \cup f(B) \subseteq f(A \cup B) \end{aligned}$$

Conversely, let $y \in f(A \cup B) \Rightarrow \exists x \in A \cup B . f(x) = y \Rightarrow x \in A$ or $x \in B \Rightarrow f(x) \in f(A)$ or $f(x) \in f(B) \Rightarrow f(x) \in f(A) \cup f(B)$. This shows that $f(A \cup B) \subseteq f(A) \cup f(B)$, hence $f(A \cup B) = f(A) \cup f(B)$.

$f^{-1}(A \cap B)$ is not properly contained in $f^{-1}(A) \cap f^{-1}(B)$, as $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$. $x \in f^{-1}(A) \cap f^{-1}(B) \Rightarrow f(x) \in A, f(x) \in B \Rightarrow f(x) \in A \cap B \Rightarrow x \in f^{-1}(A \cap B)$.

Conversely $x \in f^{-1}(A \cap B) \Rightarrow f(x) \in A \cap B \Rightarrow f(x) \in A, f(x) \in B \Rightarrow x \in f^{-1}(A), x \in f^{-1}(B) \Rightarrow x \in f^{-1}(A) \cap f^{-1}(B)$.

Thus the two sides are contained in each other, hence must be equal. ∎

Question 1(b) *Define a binary relation on a set A. Give examples of relations which are*

1. *reflexive, symmetric but not transitive.*
2. *reflexive, transitive but not symmetric.*
3. *symmetric, transitive but not reflexive.*

Solution.

1. Reflexive, symmetric but not transitive. $A = \{a, b, c\}$, and $R = \{(a, a), (b, b), (c, c), (a, b), (b, c), (b, a), (c, b)\}$.
2. reflexive, transitive but not symmetric. $A = \{a, b, c\}$, and $R = \{(a, a), (b, b), (c, c), (a, b), (b, c), (a, c)\}$.
3. symmetric, transitive but not reflexive. $A = \{a\}$, $R = \emptyset$. Note that if $(x, y) \in R$, then by symmetry $(y, x) \in R$, and then by transitivity $(x, x) \in R, (y, y) \in R$. Hence for any element $a \in A$, $(a, a) \notin R \Rightarrow (a, b) \notin R, (b, a) \notin R$ for all $b \in A$.

■

Question 2(a) *Examine whether the set of rational numbers $\{\frac{1+2m}{1+2n} \mid m, n \in \mathbb{Z}\}$ forms a group under multiplication.*

Solution. Let G be the set under consideration.

Note that the set of non-zero rational numbers forms a group under multiplication. To prove that G is its subgroup, all we need to show is that given $x, y \in G, x^{-1}y \in G$. Let $y = \frac{1+2m}{1+2n}, x = \frac{1+2q}{1+2p}$, then $x^{-1}y = \frac{1+2(p+m+2mp)}{1+2(q+n+2qn)}$. $x^{-1}y \in G$ because $m+p+2mp, q+n+2qn \in \mathbb{Z}$. Thus G is a group. ■

Question 2(b) *Show that the set of matrices*

$$\{\begin{pmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} 0 & \pm 1 \\ \pm 1 & 0 \end{pmatrix}\}$$

is a group under multiplication. Examine whether it has any proper subgroup.

Solution. Let $A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, A_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}, A_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, A_4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, A_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, A_6 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, A_7 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, A_8 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. These are all the elements of the given set G.

1. A_1 is the identity element of G.

2. G is closed w.r.t. multiplication, as shown in the following table.

	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
A_1	A_1	A_2	A_3	A_4	A_5	A_6	A_7	A_8
A_2	A_2	A_1	A_4	A_3	A_6	A_5	A_8	A_7
A_3	A_3	A_4	A_1	A_2	A_7	A_8	A_5	A_6
A_4	A_4	A_3	A_2	A_1	A_8	A_7	A_6	A_5
A_5	A_5	A_7	A_6	A_8	A_1	A_3	A_2	A_4
A_6	A_6	A_8	A_5	A_7	A_2	A_4	A_1	A_3
A_7	A_7	A_5	A_8	A_6	A_3	A_1	A_4	A_2
A_8	A_8	A_6	A_7	A_5	A_4	A_2	A_3	A_1

3. Every element of G is invertible.

4. Multiplication is associative, as it is associative for all matrices.

Thus G is a group. It has several proper subgroups — $\{A_1\}, \{A_1, A_2\}, \{A_1, A_2, A_3, A_4\}$, as is clear from the above table. ∎