UPSC Civil Services Main 1982 - Mathematics Algebra

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Mathura

Question 1(a) *Write if each of the following statements is true or false.*

1. *If $*$ is any binary operation on any set S, then $a * a = a$ for all $a \in S$.*
2. *If $*$ is any commutative binary operation on any set S, then $a * (b * c) = (b * c) * a$.*
3. *If $*$ is any associative binary operation on any set S, then $a * (b * c) = (b * c) * a$.*
4. *Every binary operation defined on a set having exactly one element is both commutative and associative.*
5. *A binary operation on a set S assigns at least one element to each ordered pair of elements of S.*
6. *A binary operation on a set S assigns at most one element to each ordered pair of elements of S.*
7. *A binary operation on a set S may associate more than one element to some ordered pair of elements of S.*
8. *A binary operation on a set S may assign exactly one element to each ordered pair of elements of S.*

Solution.

1. If $*$ is any binary operation on any set S, then $a * a = a$ for all $a \in S$.

   False. Take $S = \mathbb{Z}, * = +, 2 + 2 \neq 2$.

2. If $*$ is any commutative binary operation on any set S, then $a * (b * c) = (b * c) * a$.

   True.

3. If $*$ is any associative binary operation on any set S, then $a * (b * c) = (b * c) * a$.

   False. Take $S = 2 \times 2$ matrices, $* =$ multiplication. Let $a = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, b = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, c = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Then $a * (b * c) = a, (b * c) * a = b \neq a$.

4. Every binary operation defined on a set having exactly one element is both commutative and associative.

   True.

5. A binary operation on a set S assigns at least one element to each ordered pair of elements of S.

   True, since a binary operation assigns exactly one element to each ordered pair.

6. A binary operation on a set S assigns at most one element to each ordered pair of elements of S.

   True, same reason as above.

7. A binary operation on a set S may associate more than one element to some ordered pair of elements of S.

   False.

8. A binary operation on a set S may assign exactly one element to each ordered pair of elements of S.

   True.

■

Question 1(b) *Show that N is a normal subgroup of G if and only if $\forall g \in G.\ gNg^{-1} = N$. Does it imply that if N is a normal subgroup, then $\forall n \in N. \forall g \in G. g^{-1}ng = n$? Give an example.*

Solution. For the first part see Lemma 2.6.1, Page 50 of Algebra by Herstein.

The second statement is false. Let $G = S_3$, the symmetric group on 3 symbols. Let $N = \{\text{Identity permutation}, (123), (132)\}$, then $[G : N] = \text{Index of } N \text{ in } G = 2$, so N is normal in G.

Clearly $(123)(12) = (13)$ and $(12)(123) = (23)$. Thus $(12)^{-1}(123)(12) = (12)(123)(12) = (132) \neq (123)$. ■

Question 1(c) *Let ϕ be a homomorphism of a group G into G with kernel K. Show that $G\phi = \phi(G)$ is a group and there is an isomorphism of $G\phi$ with G/K.*

Solution. $\phi(e) = e \Rightarrow \phi(G) \neq \emptyset$. Let $y_1, y_2 \in \phi(G)$. Then there exist $x_1, x_2 \in G, \phi(x_1) = y_1, \phi(x_2) = y_2$. Thus $\phi(x_1^{-1}x_2) = \phi(x_1)^{-1}\phi(x_2) = y_1^{-1}y_2 \in \phi(G)$. Hence $\phi(G)$ is a subgroup of G, so is a group.

See Question 1(a) year 1985 for the second part. ■

Question 1(d) *Let I be an ideal in a ring R and let the coset of the element $x \in R$ be defined by $x + I = \{x + i \mid i \in I\}$. Then the distinct cosets form a partition of R, and if addition and multiplication are defined by $(x+I)+(y+I) = x+y+I, (x+I)(y+I) = xy+I$, then the cosets constitute a ring R/I in which the 0 element is $0+I$ and the inverse of $x+I$ is $(-x)+I$.*

Solution. We first show that $x + I = y + I$ if $x - y \in I$ and $x + I \cap y + I = \emptyset$ if $x - y \notin I$.

Let $j = x - y \in I$. Let $t \in x + I$. Then $t = x + i, i \in I$. $t = x + i = y + i - j \in y + I$ because $i - j \in I$, so $x + I \subseteq y + I$. Since $x - y \in I \Rightarrow y - x \in I \Rightarrow y + I \subseteq x + I$, hence $x + I = y + I$.

Conversely if $t \in (x + I) \cap (y + I)$, then $t = x + i = y + j \Rightarrow x - y = j - i \in I$. Hence if $x - y \notin I$, then $x + I \cap y + I = \emptyset$. Note that $z \in z + I$ as $0 \in I$. So the cosets form a partition of R.

1. $0 + I = I \in R/I \Rightarrow R/I \neq \emptyset$. R/I is clearly closed for addition. Note that addition is well-defined i.e. if $x + I = x_1 + I,\ y + I = y_1 + I \Rightarrow x + y + I = x_1 + y_1 + I$.

2. $(x + I) + (0 + I) = (x + 0) + I = x + I = (0 + I) + (x + I)$ for every $x \in R$, so $0 + I$ is the additive identity of R/I.

3. $(x + I) + (-x + I) = (x + (-x)) + I = 0 + I = (-x + I) + (x + I)$ so every element in R/I has an additive inverse.

4. Addition is associative and commutative, follows from the fact that this is so in R.

5. R/I is closed for multiplication. Note that multiplication is well defined — if $x + I = x_1 + I,\ y + I = y_1 + I \Rightarrow xy + I = x_1y_1 + I$, because $x = x_1 + i, y = y_1 + j,\ i, j \in I$, so $xy = (x_1 + i)(y_1 + j) = x_1y_1 + x_1j + y_1i + ij \Rightarrow xy - x_1y_1 \in I$ because $x_1j + y_1i + ij \in I$.

6. $(x + I)[(y + I) + (z + I)] = (x + I)(y + I) + (x + I)(z + I)$ as $x(y + z) = xy + xz$.

Thus R/I is a ring as desired. ∎