UPSC Civil Services Main 1984 - Mathematics Algebra

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Question 1(a) *Prove that a non-void subset S of a ring R is a subring of R if an only if $a - b \in S$ and $ab \in S$ for all $a, b \in S$.*

Solution. Let $a, b \in S$.

- $a - a = 0 \in S, 0 - b \in S$, and therefore $a - (-b) = a + b \in S$, so S is closed w.r.t. addition.
- $a + b = b + a$ as this is true in R also.
- $0 \in S$.
- We have seen $a \in S \Rightarrow -a \in S$, and $a + (-a) = 0 = (-a) + a$.
- $+$ is associative, as it is so in R. Thus S is an additive subgroup.
- $ab \in S$, given.
- Multiplication distributes over addition in S, as it does so in R.

Thus if the given requirements are satisfied, S is a subring of R. Conversely, if S is a subring of R, then $a, b \in S \Rightarrow ab \in S, -b \in S \Rightarrow a + (-b) \in S \Rightarrow a - b \in S$. ∎

Question 1(b) *Prove that an integral domain can be imbedded in a field.*

Solution. See Theorem 3.6.1 Page 140 of Algebra by Herstein. ∎

Question 1(c) *Prove that for any two ideals A and B of a ring R, the product AB is an ideal of R.*

Solution. Note first that $AB = \{\sum_{i \in F} a_i b_i \mid a_i \in A, b_i \in B, F \text{ finite}\}$.

AB is a subring of R. If $x \in AB, y \in AB$, then $x = \sum_{i \in F} a_i b_i \mid a_i \in A, b_i \in B, F$ finite and $y = \sum_{i \in G} c_i d_i \mid c_i \in A, d_i \in B, G$ finite. Clearly

$$\begin{aligned} x - y &= \sum_{i \in F} a_i b_i + \sum_{i \in G} c_i d_i \in AB \\ xy &= \Big(\sum_{i \in F} a_i b_i\Big)\Big(\sum_{i \in G} c_i d_i\Big) = \sum_{i \in H} x_i y_i \text{ where } x_i \in A, y_i \in B \end{aligned}$$

Note that $a_i b_i c_j d_j = a_i(b_i c_j) d_j$, and $a_i(b_i c_j) \in A$ as A is an ideal, $c_j \in A, b_i \in R \Rightarrow b_i c_j \in A$. Thus xy can be expressed as above.

For any $r \in R, xr = \sum_{i \in F} a_i(b_i r) = \sum_{i \in F} a_i b_i', b_i' \in B$ as B is an ideal. So $xr \in AB$. Similarly $rx \in AB$. Thus AB is an ideal. ∎