UPSC Civil Services Main 1985 - Mathematics Algebra

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Mathura

Question 1(a) *State and prove the fundamental theorem of homomorphisms of groups.*

Solution. Theorem. Let f be a homomorphism from a group G onto a group G' with kernel N. Then N is a normal subgroup of G, and G/N is isomorphic to G'.

Proof: $N = \{x \mid x \in G, f(x) = e'\}$, where e' is the identity of G'. Clearly $N \neq \emptyset$, as $e \in N$. If $x, y \in N$, then $f(x) = e', f(y) = e'$, so $f(xy^{-1}) = f(x)f(y)^{-1} = e' \Rightarrow xy^{-1} \in N \Rightarrow N$ is a subgroup of G.

Now let $g \in N$, then for any $x \in G$, $f(xgx^{-1}) = f(x)e'f(x)^{-1} = e' \Rightarrow xgx^{-1} \in N$, so N is a normal subgroup of G.

Let $\phi : G/N \longrightarrow G'$ defined by $\phi(gN) = f(g)$.

- ϕ is well defined: We need to show that ϕ does not depend on the choice of coset representative i.e. if $g_1N = g_2N$ then $\phi(g_1N) = \phi(g_2N)$. Now $g_1N = g_2N \Rightarrow g_2^{-1}g_1 \in N \Rightarrow f(g_2^{-1}g_1) = e' \Rightarrow f(g_1) = f(g_2) \Rightarrow \phi(g_1N) = \phi(g_2N)$.

- ϕ is a homomorphism: $\phi(g_1N)\phi(g_2N) = f(g_1)f(g_2) = f(g_1g_2) = \phi(g_1g_2N)$.

- ϕ is onto: Let $y \in G'$, then f being onto, there exists $x \in G$ such that $f(x) = y$. Clearly $\phi(xN) = f(x) = y$.

- ϕ is 1-1. If $g_1N \neq g_2N$, then $g_2^{-1}g_1 \neq N$, so $f(g_2^{-1}g_1) \neq e' \Leftrightarrow f(g_1) \neq f(g_2) \Leftrightarrow \phi(g_1N) \neq \phi(g_2N)$.

Thus ϕ is an isomorphism from G/N onto G', so $G/N \simeq G'$.

Note 1: If $\eta : G \longrightarrow G/N$ is the natural homomorphism i.e. $\eta(g) = gN$, then $f = \phi \circ \eta$.

Note 2: If f is not assumed to be onto, we can only say that $G/N \simeq f(G)$. ∎

Question 1(b) *Prove that the order of each subgroup of a finite group divides the order of the group.*

Solution. See Theorem 2.4.1 Page 41 of Algebra by Herstein. ∎

Question 1(c) *Is each of the following statements true or false?*

1. *If a is an element of a ring $(R, +, .)$ and $m, n \in \mathbb{Z}$, then $(a^m)^n = a^{mn}$.*
2. *Every subgroup of an abelian group is not necessarily abelian.*
3. *A semigroup $(G, .)$ is which the equations $ya = b, ax = b$ are solvable for any a, b, is a group.*
4. *The relation of isomorphism in the class of all groups is not an equivalence relation.*
5. *There are only two abstract groups of order 6.*

Solution.

1. If a is an element of a ring $(R, +, .)$ and $m, n \in \mathbb{Z}$, then $(a^m)^n = a^{mn}$.

   True. $(a^m)^n = \underbrace{a^m \ldots . a^m}_{n \text{ times}} = \underbrace{a.a.a.a. \ldots . a}_{mn \text{ times}} = a^{mn}$.

2. Every subgroup of an abelian group is not necessarily abelian.

   False. If G is abelian, and H is a subgroup of G, then for any $a, b \in H$, $a, b \in G \Rightarrow ab = ba \Rightarrow H$ is abelian.

3. A semigroup $(G, .)$ is which the equations $ya = b, ax = b$ are solvable for any a, b, is a group.

   True. Let $a \in G$, then $ax = a$ is solvable $\Rightarrow$ there exists an element $e \in G$ such that $ae = a$. Now let $b \in G$, then there exists $y \in G, ya = b$. Then $be = yae = ya = b$. Thus e is the right identity for G. For any $a \in G$, the equation $ax = e$ is solvable, thus a has a right inverse.

   Since G has a right identity and a every element of G has a right inverse, G is a group. Note that since G is a semigroup, it is already closed the semigroup operation, and the operation is associative.

4. The relation of isomorphism in the class of all groups is not an equivalence relation.

   False.

   - The relation $\simeq$ is reflexive, as the identity map is an isomorphism from any group to itself.

- If $\sigma : G \longrightarrow G'$ is an isomorphism, then $\sigma^{-1} : G' \longrightarrow G$ is an isomorphism, so $\simeq$ is symmetric.
- If $G_1 \simeq G_2$ and $\sigma_1 : G_1 \longrightarrow G_2$, and $G_2 \simeq G_3$, $\sigma_2 : G_2 \longrightarrow G_3$, then $\sigma_2 \circ \sigma_1 : G_1 \longrightarrow G_3$ is also an isomorphism, so $G_1 \simeq G_3$, thus $\simeq$ is transistive.

5. There are only two abstract groups of order 6.

   True. The two groups are the cyclic group of order 6 and S_3, the symmetric group on 3 symbols.

   If G is abelian, then G has x, an element of order 2 and y, an element of order 3. Since $xy = yx$, $o(xy) = 6$ so G is cyclic.

   If G is non-abelian, let $a, b \in G$, where $o(a) = 2, o(b) = 3$ (such elements exist because of Cauchy's theorem), then $G = \{e, a, b, b^2, ab, ba\} \simeq S_3$.

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