UPSC Civil Services Main 1986 - Mathematics Algebra

Brij Bhooshan

Asst. Professor

B.S.A. College of Engg & Technology

Mathura

Question 1(a) *Prove that a map $f : X \longrightarrow Y$ is injective if and only if f can be left cancelled in the sense that $f \circ g = f \circ h \Rightarrow g = h$. Prove that f is surjective if and only if it can be right cancelled in the sense that $g \circ f = h \circ f \Rightarrow g = h$.*

Solution. Suppose that f is injective (1-1). Let $f \circ g = f \circ h$, where g, h are mappings from $Z \longrightarrow X$, where Z can be arbitrary. We have to prove $g(z) = h(z)\ \forall z \in Z$. If $g(z) \neq h(z)$ then $f(g(z)) \neq f(h(z))$, since f is injective, consequently $f \circ g \neq f \circ h$. Thus $g(z) = h(z) \forall z \in Z$, so $f \circ g = f \circ h \Rightarrow g = h$.

Conversely let $f \circ g = f \circ h \Rightarrow g = h$. We have to prove that f is 1-1. Suppose it is not 1-1. Then there exist $x_1, x_2 \in X, x_1 \neq x_2, f(x_1) = f(x_2)$. Define $g, h : X \longrightarrow X$ by $g(x) = x_1, h(x) = x_2$ for all $x \in X$. Then $\forall x \in X\ f(g(x)) = f(x_1) = f(x_2) = f(h(x))$, so $f \circ g = f \circ h$, but $g \neq h$, which is a contradiction. Thus f must be 1-1.

Suppose that f is onto. Given $(g \circ f)(x) = (h \circ f)(x)$, we need to show that $g(y) = h(y)$ for all $y \in Y$. For any $y \in Y$, there exists $x \in X$ such that $f(x) = y$, since f is onto. Thus $g(y) = (g \circ f)(x) = (h \circ f)(x) = h(y)$, showing that $g \circ f = h \circ f \Rightarrow g = h$.

Conversely let $g \circ f = h \circ f \Rightarrow g = h$. We have to prove that f is onto. Suppose it is not onto. Then there exists $y_0 \in Y$ such that there is no $x \in X$ such that $f(x) = y_0$. Let $g, h : Y \longrightarrow Z$ (for any Z) be defined by $g(y) = h(y)$ if $y \neq y_0$, and $g(y_0) \neq h(y_0)$. Clearly $(g \circ f)(x) = (g \circ f)(x)$ for every $x \in X$, but $g \neq h$, which is a contradiction. Thus f is onto. ∎

Question 1(b) *The product HK of two subgroups H, K of a group G is a subgroup of G if and only if $HK = KH$.*

Solution. See Lemma 2.5.1 Page 44 of Algebra by Herstein. ∎

Question 1(c) *Prove that a finite integral domain is a field.*

Solution. See lemma 3.2.2 Page 127 of Algebra by Herstein. ∎