UPSC Civil Services Main 1987 - Mathematics Algebra

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Question 1(a) *Let $f : X \longrightarrow Y, g : Y \longrightarrow Z$ be bijections, prove that $g \circ f$ is a bijection and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

Solution. $g \circ f : X \longrightarrow Z$. Let $x_1 \neq x_2$ then then $f(x_1) \neq f(x_2)$ because f is 1-1. Thus $g(f(x_1)) \neq g(f(x_2))$ because g is 1-1. Thus $g \circ f$ is 1-1.

For any $z \in Z, \exists y \in Y$ such that $g(y) = z$, because g is onto. Now $y \in Y$, so there exists $x \in X$ such that $f(x) = y$, because f is onto. Thus for any $z \in Z$, we have determined $x \in X$ such that $(g \circ f)(x) = g(f(x)) = g(y) = z$, thus $g \circ f$ is onto. Thus $g \circ f$ is invertible.

$$\begin{aligned}(g \circ f) \circ (f^{-1} \circ g^{-1})(z) &= (g \circ f)f^{-1}(g^{-1}(z)) \\ &= (g \circ f)(f^{-1}(g^{-1}(z))) \\ &= g(f(f^{-1}(g(z)))) \\ &= g(g^{-1}(z)) = z \quad \forall z \in Z\end{aligned}$$

Thus $(g \circ f) \circ (f^{-1} \circ g^{-1}) = I_Z$.

$$\begin{aligned}(f^{-1} \circ g^{-1}) \circ (g \circ f)(x) &= (f^{-1} \circ g^{-1})(g(f(x)) \\ &= f^{-1}(g^{-1}(g(f(x)))) \\ &= f^{-1}(f(x)) = x \quad \forall x \in X\end{aligned}$$

Thus $(f^{-1} \circ g^{-1}) \circ (g \circ f) = I_X$, so $f^{-1} \circ g^{-1} = g \circ f$. ∎

Question 1(b) *Prove that $H * K$ is a subgroup of $(G, *)$ if and only if $H * K = K * H$.*

Solution. See Lemma 2.5.1, Page 44 of Algebra by Herstein. ∎

Question 1(c) *If G is a finite group of order g and H is a subgroup of G of order h , then prove that h is a factor of g.*

Solution. See Theorem 2.4.1, Page 41 of Algebra by Herstein. ∎