UPSC Civil Services Main 1979 - Mathematics
Complex Analysis

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Question 1(a) *If a function $f(z)$ is analytic and bounded in the whole plane, show that $f(z)$ reduces to a constant. Hence show that every polynomial has a root.*

Solution. See 1989, question 2(b) for the first part. See 1996 question 2(a) for the second part. ∎

Question 1(b) *Evaluate the following integrals by the method of residues.*

1.

$$\int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\, d\theta \qquad (a > b > 0)$$

2.

$$\int_0^{\infty} \frac{x^{\frac{1}{6}}\log x}{(1+x)^2}\, dx$$

Solution.

1. Let

$$I = \int_0^{2\pi} \frac{\sin^2\theta}{a + b\cos\theta}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1-\cos 2\theta}{a + b\cos\theta}\, d\theta$$

Let $I_1 = \dfrac{1}{2}\displaystyle\int_0^{2\pi} \frac{1}{a + b\cos\theta}\, d\theta$. Put $z = e^{i\theta}$ so that

$$I_1 = \frac{1}{2}\int_{|z|=1} \frac{dz}{iz(a + \frac{b}{2}(z + \frac{1}{z}))} = \frac{1}{i}\int_{|z|=1} \frac{dz}{bz^2 + 2az + b}$$

The integrand $\dfrac{1}{bz^2+2az+b}$ has two simple poles at $z_1 = \dfrac{-a+\sqrt{a^2-b^2}}{b}$, $z_2 = \dfrac{-a-\sqrt{a^2-b^2}}{b}$. Since $a > b > 0$, $|z_2| > 1$, but $|z_1 z_2| = 1$ so $|z_1| < 1$ i.e. the pole at $z = z_1$ lies within $|z| \le 1$.

Residue at z_1 is $\lim\limits_{z\to z_1} \dfrac{z-z_1}{bz^2+2az+b} = \dfrac{1}{2bz_1+2a} = \dfrac{1}{2\sqrt{a^2-b^2}}$. Thus $I_1 = 2\pi i \dfrac{1}{i}\dfrac{1}{2\sqrt{a^2-b^2}} = \dfrac{\pi}{\sqrt{a^2-b^2}}$. Let

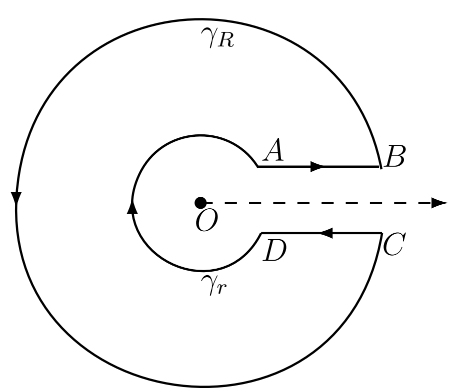
$$
\begin{aligned}
I_2 &= \frac{1}{2}\int_0^{2\pi}\frac{\cos 2\theta}{a+b\cos\theta}\,d\theta \;=\; \frac{1}{2}\,\mathrm{Re}\int_0^{2\pi}\frac{e^{2i\theta}\,d\theta}{a+b\cos\theta}\\
&= \frac{1}{2}\,\mathrm{Re}\,\frac{1}{i}\int_{|z|=1}\frac{2z^2\,dz}{bz^2+2az+b}\\
&= \mathrm{Re}\,\frac{1}{i}\times 2\pi i \text{ Residue of } \frac{z^2}{bz^2+2az+b} \text{ at } z=z_1\\
&= 2\pi\frac{1}{b}\frac{z_1^2}{z_1-z_2}
\end{aligned}
$$

Thus

$$
\begin{aligned}
I_1 - I_2 &= \frac{2\pi}{b(z_1-z_2)} - \frac{2\pi z_1^2}{b(z_1-z_2)}\\
&= \frac{2\pi}{2\sqrt{a^2-b^2}}(1-z_1^2)\\
&= \frac{\pi}{\sqrt{a^2-b^2}}\Big(1 - \frac{a^2-2a\sqrt{a^2-b^2}+(a^2-b^2)}{b^2}\Big)\\
&= \frac{\pi}{\sqrt{a^2-b^2}}\Big(2\sqrt{a^2-b^2}\frac{a-\sqrt{a^2-b^2}}{b^2}\Big)
\end{aligned}
$$

Thus $I = \dfrac{2\pi}{a+\sqrt{a^2-b^2}}$.

2. Let $f(z) = \frac{z^{\frac{1}{6}}\log z}{(1+z)^2}$ and the contour C as shown. γ_r is a circle of radius r oriented clockwise, and γ_R a circle of radius R oriented anticlockwise. AB is along x-axis on which $z = x$, CD is the line on which $z = xe^{2\pi i}$. To avoid the branch point of the multiple valued function $\log z$, we consider $\mathbb{C}-$ positive side of the x-axis. We choose the branch of $\log z$ for which $\log z = \log|z| + i\theta, 0 < \theta \le 2\pi$.



(a) Clearly $f(z)$ has a double pole at $z=-1$. Residue of $f(z)$ at $z=-1$ is

$$\begin{aligned}
& \frac{1}{1!}\frac{d}{dz}\left[\frac{(z+1)^2 z^{\frac{1}{6}}\log z}{(z+1)^2}\right]_{at\ \ z=1} \\
= & \left[\frac{z^{\frac{1}{6}}}{z}+\frac{1}{6}z^{-\frac{5}{6}}\log z\right]_{at\ \ z=-1=e^{i\pi}} = \frac{\log z+6}{6z^{\frac{5}{6}}}\ at\ z=e^{i\pi} \\
= & \frac{\log e^{i\pi}+6}{6e^{\frac{5i\pi}{6}}} = \frac{i\pi+6}{6}\left(\cos\frac{5\pi}{6}-i\sin\frac{5\pi}{6}\right) \\
= & \frac{i\pi+6}{6}\left(-\frac{\sqrt{3}}{2}-\frac{1}{2}i\right) = -\frac{1}{12}(6+i\pi)(\sqrt{3}+i)
\end{aligned}$$

(b) On γ_R, $z=Re^{i\theta}$, $|z+1|\geq|z|-1=R-1$ and $|\log z|=|\log Re^{i\theta}|=|\log R+i\theta|\leq \log R+\theta\leq\log R+2\pi$ as $0\leq\theta\leq2\pi$. Thus

$$\left|\int_{\gamma_R}\frac{z^{\frac{1}{6}}\log z}{(1+z)^2}\,dz\right|\leq\int_0^{2\pi}\frac{R^{\frac{1}{6}}(\log R+2\pi)}{(R-1)^2}R\,d\theta=2\pi\frac{R^{\frac{7}{6}}}{(R-1)^2}(\log R+2\pi)$$

Clearly $\lim\limits_{R\to\infty}\left[\frac{R^{\frac{7}{6}}\log R}{(R-1)^2}+\frac{2\pi R^{\frac{7}{6}}}{(R-1)^2}\right]=0$, and therefore

$$\lim_{R\to\infty}\int_{\gamma_R}\frac{z^{\frac{1}{6}}\log z}{(1+z)^2}\,dz=0$$

(c) On γ_r, $z=re^{i\theta}$, $|z+1|\geq1-|z|=1-r$ and $|\log z|=|\log re^{i\theta}|=|\log r+i\theta|\leq \log r+\theta\leq\log r+2\pi$ as $0\leq\theta\leq2\pi$. Thus

$$\left|\int_{\gamma_r}\frac{z^{\frac{1}{6}}\log z}{(1+z)^2}\,dz\right|\leq\int_0^{2\pi}\frac{r^{\frac{1}{6}}(\log r+2\pi)}{(1-r)^2}r\,d\theta=2\pi\frac{r^{\frac{7}{6}}}{(1-r)^2}(\log r+2\pi)$$

But $\lim\limits_{r\to0}\left[\frac{r^{\frac{7}{6}}\log r}{(1-r)^2}+\frac{2\pi r^{\frac{7}{6}}}{(1-r)^2}\right]=0$, and therefore

$$\lim_{r\to0}\int_{\gamma_r}\frac{z^{\frac{1}{6}}\log z}{(1+z)^2}\,dz=0$$

By Cauchy's residue theorem, using $1,2,3$, we get

$$\lim_{\substack{R\to\infty\\ r\to0}}\int_C f(z)\,dz=\int_0^\infty\frac{x^{\frac{1}{6}}\log x}{(1+x)^2}\,dx+\int_\infty^0\frac{(xe^{2\pi i})^{\frac{1}{6}}\log(xe^{2\pi i})}{(1+x)^2}\,dx$$

because on AB, $z=x$ and on CD, $z=xe^{2\pi i}$. Therefore

$$\begin{aligned}
\int_0^\infty\frac{x^{\frac{1}{6}}\log x}{(1+x)^2}\,dx-\int_0^\infty\frac{x^{\frac{1}{6}}e^{\frac{2\pi i}{6}}(\log x+2\pi i)}{(1+x)^2}\,dx &= -\frac{2\pi i}{12}(6+i\pi)(\sqrt{3}+i) \\
\Rightarrow\left(\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\int_0^\infty\frac{x^{\frac{1}{6}}\log x}{(1+x)^2}\,dx-\int_0^\infty\frac{x^{\frac{1}{6}}(\frac{1}{2}+\frac{\sqrt{3}}{2}i)2\pi i}{(1+x)^2}\,dx &= -\frac{\pi}{6}\Big[-(6+\pi\sqrt{3})+i(6\sqrt{3}-\pi)\Big]
\end{aligned}$$

Equating real and imaginary parts, we get

$$\frac{1}{2}\int_0^\infty \frac{x^{\frac{1}{6}}\log x}{(1+x)^2}\,dx + \sqrt{3}\pi\int_0^\infty \frac{x^{\frac{1}{6}}}{(1+x)^2}\,dx = \frac{\pi}{6}(6+\pi\sqrt{3}) \qquad (1)$$

$$-\frac{\sqrt{3}}{2}\int_0^\infty \frac{x^{\frac{1}{6}}\log x}{(1+x)^2}\,dx - \pi\int_0^\infty \frac{x^{\frac{1}{6}}}{(1+x)^2}\,dx = \frac{\pi}{6}(\pi-6\sqrt{3}) \qquad (2)$$

Multiplying (1) by $\sqrt{3}$ and adding

$$-\int_0^\infty \frac{x^{\frac{1}{6}}\log x}{(1+x)^2}\,dx = \frac{\pi}{6}\left[6+\pi\sqrt{3}+\sqrt{3}\pi-18\right] = \frac{\pi}{6}\left[2\pi\sqrt{3}-12\right]$$

Thus

$$\int_0^\infty \frac{x^{\frac{1}{6}}\log x}{(1+x)^2}\,dx = 2\pi - \frac{\pi^2}{\sqrt{3}}$$

In addition, multiplying (2) by $\sqrt{3}$ and adding, we get

$$2\pi\int_0^\infty \frac{x^{\frac{1}{6}}}{(1+x)^2}\,dx = \frac{\pi}{6}\left[6\sqrt{3}+3\pi+\pi-6\sqrt{3}\right]$$

giving us

$$\int_0^\infty \frac{x^{\frac{1}{6}}}{(1+x)^2}\,dx = \frac{2\pi}{3}$$

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