UPSC Civil Services Main 1980 - Mathematics Complex Analysis

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Mathura

Question 1(a) *Find the expansion in powers of z of* $\dfrac{1}{z(z-1)(z+3)}$ *in the region* $0 < |z| < 4$.

Solution. It can easily be seen that

$$f(z) = \frac{1}{z(z-1)(z+3)} = -\frac{1}{3z} + \frac{1}{4(z-1)} + \frac{1}{12(z+3)}$$

1. Region $0 < |z| < 1$.

$$\begin{aligned} f(z) &= -\frac{1}{3z} - \frac{1}{4}(1-z)^{-1} + \frac{1}{36}\Big(1 + \frac{z}{3}\Big)^{-1} \\ &= -\frac{1}{3z} - \frac{1}{4}\sum_{n=0}^{\infty} z^n + \frac{1}{36}\sum_{n=0}^{\infty}\Big(\frac{z}{3}\Big)^n(-1)^n \\ &= -\frac{1}{3z} - \frac{2}{9} + \sum_{n=1}^{\infty} z^n\Big(-\frac{1}{4} + \frac{(-1)^n}{3^n}\Big) \end{aligned}$$

   This is the Laurent expansion of $f(z)$ in the region $0 < |z| < 1$. The given function satisfies the requirements of Laurent's theorem.

2. Region $1 < |z| < 3$.

$$\begin{aligned} f(z) &= -\frac{1}{3z} + \frac{1}{4z}\Big(1 - \frac{1}{z}\Big)^{-1} + \frac{1}{36}\Big(1 + \frac{z}{3}\Big)^{-1} \\ &= -\frac{1}{3z} + \frac{1}{4z}\sum_{n=0}^{\infty}\frac{1}{z^n} + \frac{1}{36}\sum_{n=0}^{\infty}\Big(\frac{z}{3}\Big)^n(-1)^n \\ &= \frac{1}{36}\sum_{n=0}^{\infty}(-1)^n\Big(\frac{z}{3}\Big)^n - \frac{1}{12z} + \frac{1}{4}\sum_{n=1}^{\infty}\frac{1}{z^{n+1}} \end{aligned}$$

This is again the Laurent expansion valid in the annular region $1 < |z| < 3$.

3. Region $|z| > 3$

$$\begin{aligned} f(z) &= -\frac{1}{3z} + \frac{1}{4z}\Big(1 - \frac{1}{z}\Big)^{-1} + \frac{1}{3z}\Big(1 + \frac{3}{z}\Big)^{-1} \\ &= -\frac{1}{3z} + \frac{1}{4z}\sum_{n=0}^{\infty}\frac{1}{z^n} + \frac{1}{3z}\sum_{n=0}^{\infty}\Big(\frac{3}{z}\Big)^n(-1)^n \\ &= \frac{1}{4z} + \sum_{n=1}^{\infty}\frac{1}{z^{n+1}}\Big(\frac{1}{4} + 3^{n-1}(-1)^n\Big) \end{aligned}$$

The is Taylor's expansion of $f(z)$ around ∞.

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Question 1(b) *Evaluate by contour integration*

1. $\displaystyle\int_0^{\infty}\frac{dx}{x^4+1}$

2. $\displaystyle\int_0^{2\pi}\frac{\cos 2\theta}{5+4\sin\theta}\,d\theta$

Solution.

1. See 2001 question 2(b).

2. The given integral is the real part of

$$I = \int_0^{2\pi}\frac{e^{2i\theta}\,d\theta}{5+4\sin\theta}$$

Put $z = e^{i\theta}, dz = ie^{i\theta}\,d\theta$ so that

$$I = \int_{|z|=1}\frac{z^2}{5+\frac{4}{2i}(z-\frac{1}{z})}\frac{dz}{iz} = \int_{|z|=1}\frac{z^2\,dz}{5iz+2z^2-2}$$

The integrand $\dfrac{z^2}{5iz+2z^2-2}$ has two simple poles, which are given by $2z^2+5iz-2=0$ or $2(z+\frac{i}{2})(z+2i)=0$. Out of the two poles $z=-2i, -\frac{i}{2}$, only $z=-\frac{i}{2}$ is inside the unit disc $|z|\le 1$. Residue at this pole is given by $\dfrac{(\frac{i}{2})^2}{2(-\frac{i}{2}+2i)} = \frac{i^2}{4(3i)} = \frac{i}{12}$. Thus by Cauchy's residue theorem

$$\int_{|z|=1}\frac{z^2\,dz}{5iz+2z^2-2} = \int_0^{2\pi}\frac{e^{2i\theta}}{5+4\sin\theta}\,d\theta = 2\pi i\frac{i}{12} = -\frac{\pi}{6}$$

Equating real and imaginary parts, we get

$$\int_0^{2\pi} \frac{\cos 2\theta}{5 + 4\sin\theta}\, d\theta = -\frac{\pi}{6}, \quad \int_0^{2\pi} \frac{\sin 2\theta}{5 + 4\sin\theta}\, d\theta = 0$$

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