UPSC Civil Services Main 1981 - Mathematics
Complex Analysis

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Mathura

Question 1(a) *State and prove Cauchy's integral formula.*

Solution. See 1986 question 1(a). ∎

Question 1(b) *Evaluate*

1. $\displaystyle\int_0^\infty \frac{x^{-k}}{x+1}\,dx,\ 0 < k < 1.$
2. $\displaystyle\int_0^\infty \frac{\sin^2 x}{x^2}\,dx$

Solution.

1. We shall show that for $0 < a < 1$

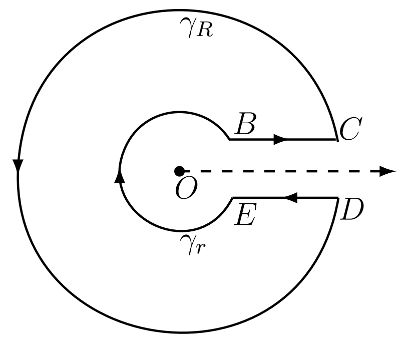
$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = \frac{\pi}{\sin a\pi}$$

Now let $a-1 = -k$ so that $a = 1-k$ and $0 < a < 1 \Leftrightarrow 0 < k < 1$. Thus for $0 < k < 1$,

$$\int_0^\infty \frac{x^{-k}}{x+1}\,dx = \frac{\pi}{\sin(1-k)\pi} = \frac{\pi}{\sin k\pi}$$

We consider first of all $\mathbb{C} - \{\text{positive real axis}\}$ i.e. there is a cut along the real axis for which $x \geq 0$ to make $\log z$ single valued. We choose that branch of $\log z$ for which $\log z = \log x$ when $z = x, x > 0$.

For $\int_0^\infty \frac{x^{a-1}}{1+x}\,dx$ we take $f(z) = \frac{z^{a-1}}{1+z}$ and the contour C as shown in the figure. γ_r is a circle of radius r oriented clockwise, and γ_R is a circle of radius R oriented anticlockwise. BC is the line joining $(r,0)$ to $(R,0)$, so is DE. We finally make $r \to 0$ and $R \to \infty$. Note that on BC $z^{a-1} = x^{a-1}$ and on DE $z^{a-1} = (xe^{2\pi i})^{a-1}$.



(a) Clearly $f(z) = \dfrac{z^{a-1}}{1+z}$ has a simple pole at $z=-1$ inside the contour. Residue at $z=-1=e^{\pi i}$ of $f(z)$ is $(e^{\pi i})^{a-1}$. Thus

$$\lim_{\substack{R\to\infty\\ r\to 0}} \int_C \frac{z^{a-1}}{1+z}\,dz = 2\pi i(-e^{\pi i a})$$

Note that $z=0$ is excluded by the cut.

(b)
$$\left|\int_{\gamma_R} \frac{z^{a-1}}{1+z}\,dz\right| = \left|\int_0^{2\pi} \frac{R^{a-1}e^{i\theta(a-1)}}{1+Re^{i\theta}} Rie^{i\theta}\,d\theta\right| \leq \frac{R^{a-1}R}{R-1}2\pi$$

Here we use $|z+1| \geq |z|-1$. Thus $\displaystyle\lim_{R\to\infty} \int_{\gamma_R} \frac{z^{a-1}}{1+z}\,dz = 0$ as $a<1$.

(c) Similarly
$$\left|\int_{\gamma_r} \frac{z^{a-1}}{1+z}\,dz\right| \leq \frac{r^a}{1-r}2\pi$$

because $|z+1| \geq 1-|z|$. Thus $\displaystyle\lim_{r\to 0} \int_{\gamma_r} \frac{z^{a-1}}{1+z}\,dz = 0$.

Thus

$$\begin{aligned}\lim_{\substack{R\to\infty\\ r\to 0}} \int_C \frac{z^{a-1}}{1+z}\,dz &= \lim_{\substack{R\to\infty\\ r\to 0}} \int_{BC} \frac{x^{a-1}}{1+x}\,dx + \lim_{\substack{R\to\infty\\ r\to 0}} \int_{DE} \frac{x^{a-1}e^{2\pi i(a-1)}}{1+x}\,dx \\ &= \int_0^\infty \frac{x^{a-1}}{1+x}\,dx + \int_\infty^0 \frac{x^{a-1}}{1+x}e^{2\pi i a}\,dx\end{aligned}$$

as on BC, $z=x$ and on DE, $z=xe^{2\pi i}$. Thus

$$\int_0^\infty \frac{x^{a-1}}{1+x}(1-e^{2\pi i a})\,dx = -2\pi i e^{\pi i a}$$

or

$$\int_0^\infty \frac{x^{a-1}}{1+x}\,dx = -2\pi i \frac{e^{\pi i a}}{1-e^{2\pi i a}} = \pi \frac{-2i}{e^{-\pi i a}-e^{\pi i a}} = \frac{\pi}{\sin a\pi}$$

Alternate proof: This avoids the use of multiple valued functions. In 1991, question 2(c), we proved

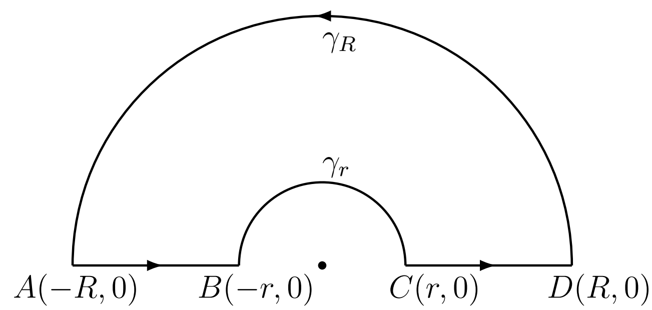
$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{\pi}{\sin a\pi} \text{ for } 0 < a < 1$$

Put $e^x = t$, then

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \int_0^{\infty} \frac{t^a}{1+t}\,\frac{dt}{t} = \int_0^{\infty} \frac{t^{a-1}}{1+t}\,dt$$

Thus $\displaystyle\int_0^{\infty} \frac{t^{a-1}}{1+t}\,dt = \frac{\pi}{\sin a\pi}$, $0 < a < 1$.

2. Clearly $\displaystyle\int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx$ and $\dfrac{\sin^2 x}{x^2}$ is the real part of $\dfrac{1-e^{2ix}}{2x^2}$, therefore we take $f(z) = \dfrac{1-e^{2iz}}{2z^2}$ and the contour C as shown. Finally we let $R \to \infty, r \to 0$.



(a) On γ_R, $z = Re^{i\theta}$ and

$$|1-e^{2iz}| = |1-e^{2i(R\cos\theta+iR\sin\theta)}| \leq 1 + |e^{2i(R\cos\theta+iR\sin\theta)}| \leq 2$$

because $|e^{2iR\cos\theta}| = 1$ and $|e^{-2R\sin\theta}| \leq 1$ as $\sin\theta > 0$ for $0 < \theta < \pi$. Therefore

$$\left|\int_{\gamma_R} \frac{1-e^{2iz}}{2z^2}\,dz\right| \leq \frac{2}{2R^2}\pi R = \frac{\pi}{R}$$

and hence $\displaystyle\lim_{R\to\infty}\int_{\gamma_R} \frac{1-e^{2iz}}{2z^2}\,dz = 0$.

(b) Residue of $f(z)$ at $z = 0$: Note that $z = 0$ is a simple pole, so the residue is $\displaystyle\lim_{z\to 0} z\frac{1-e^{2iz}}{2z^2} = \lim_{z\to 0}\frac{1-e^{2iz}}{2z} = \lim_{z\to 0}\frac{-2ie^{2iz}}{2} = -i$. Thus

$$\lim_{r\to 0}\int_{\gamma_r} \frac{1-e^{2iz}}{2z^2}\,dz = i(-i)(0-\pi) = -\pi$$

Here we have used the following property: If $f(z)$ has a simple pole at $z = a$ and γ_r is a circular arc (part of a circle with center a and radius r), from θ_1 to θ_2, then

$$\lim_{r\to 0}\int_{\gamma_r} f(z)\,dz = ia_{-1}(\theta_2 - \theta_1)$$

where a_{-1} is the residue of $f(z)$ at $z = a$. See 1985, question 1(c) for more details and proof.

Thus

$$\lim_{\substack{R\to\infty\\ r\to 0}} \int_C \frac{1-e^{2\pi iz}}{2z^2}\,dz = \int_{-\infty}^{\infty} \frac{1-e^{2\pi ix}}{2x^2}\,dx - \pi = 0$$

as there is no singularity inside C. Taking real parts, we get

$$\int_{-\infty}^{\infty} \frac{\sin^2 x}{x^2}\,dx = \pi \Longrightarrow \int_0^{\infty} \frac{\sin^2 x}{x^2}\,dx = \frac{\pi}{2}$$

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Question 1(c) *Obtain the Laurent expansion in powers of z of*

$$z + \frac{1}{z-1} + \frac{\sinh z}{z^2}$$

Solution.

1. $\frac{1}{z-1}$ is analytic in the annular region $0 \leq |z| < 1$, so we have the Taylor series for $\frac{1}{z-1}$ valid in $0 \leq |z| < 1$. In fact for $|z| < 1$,

$$\frac{1}{z-1} = -(1-z)^{-1} = -\sum_{n=0}^{\infty} z^n$$

2. $\frac{\sinh z}{z^2}$ has a simple pole at $z = 0$ and is analytic everywhere else. We have Laurent series valid in $|z| > 0$:

$$\frac{\sinh z}{z^2} = \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

Note that $\sinh z = \dfrac{e^z + e^{-z}}{2}$, which gives us the desired expansion.

Thus

$$z + \frac{1}{z-1} + \frac{\sinh z}{z^2} = z - \sum_{n=0}^{\infty} z^n + \frac{1}{z^2}\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}$$

or

$$z + \frac{1}{z-1} + \frac{\sinh z}{z^2} = \frac{1}{z} - 1 + \frac{z}{3!} + \sum_{n=1}^{\infty} z^{2n+1}\left(\frac{1}{(2n+3)!} - 1\right) - \sum_{n=1}^{\infty} z^{2n}$$

and this expansion is valid in $0 < |z| < 1$. ∎