UPSC Civil Services Main 1982 - Mathematics Complex Analysis

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Mathura

Question 1(a) *Evaluate by contour integration*

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx, \quad 0 < a < 1$$

Solution. See 1991, question 2(c). ∎

Question 1(b) *Find the function $f(z)$, holomorphic within the unit circle, which takes the values*

$$\frac{a - \cos\theta + i\sin\theta}{a^2 - 2a\cos\theta + 1}$$

on the circle.

Solution. See 1997, question 2(c). ∎

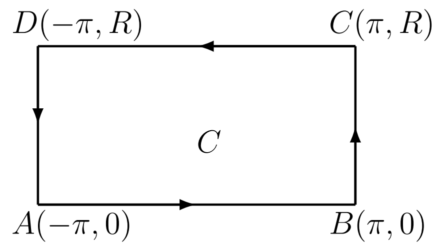
Question 1(c) *Find by contour integration the value of*

$$\int_0^{\pi} \frac{x\sin x\,dx}{a^2 - 2a\cos x + 1}$$

if $a > 1$.

Solution. Note: Even though the question restricts us to $a > 1$, we shall also consider the case $0 < a < 1$ for completeness.

We take $f(z) = \frac{z}{1-ae^{-iz}}$ and the contour C is the rectangle $ABCD$ where $A = (-\pi, 0), B = (\pi, 0), C = (\pi, R), D = (-\pi, R)$ oriented in the anticlockwise direction. We let $R \to \infty$ eventually.



Clearly $f(z)$ has simple poles at points z given by $e^{iz} = a = e^{\log a + 2n\pi i}$, $n \in \mathbb{Z}$. Thus $z = -i \log a + 2n\pi, n \in \mathbb{Z}$.

Thus when $a > 1$ i.e. $\log a > 0$, $f(z)$ has no pole in the vertical strip bounded by $x = -\pi, x = \pi, y > 0$. When $0 < a < 1$, $f(z)$ has a simple pole at $z = -i \log a$ inside C.

Residue at $z = -i \log a$ is given by $\lim_{z \to -i \log a} \frac{(z + i \log a) z}{1 - ae^{-iz}} = \frac{-i \log a}{aie^{-\log a}} = -\log a$.

Thus by Cauchy's residue theorem,

$$\lim_{R \to \infty} \int_C \frac{z}{1 - ae^{-iz}}\, dz = \begin{cases} 0, & \text{when } a > 1 \\ -2\pi i \log a, & \text{when } 0 < a < 1 \end{cases}$$

1. On CD, $z = x + iR$, x varies from π to $-\pi$.

$$\left| \int_{CD} \frac{z}{1 - ae^{-iz}}\, dz \right| = \left| \int_{\pi}^{-\pi} \frac{x + iR}{1 - ae^{-i(x+iR)}}\, dx \right|$$

Now $|x + iR| \le |x| + |R|, |1 - ae^{-ix}e^R| \ge |ae^R e^{-ix}| - 1$ and therefore

$$\left| \int_{CD} \frac{z}{1 - ae^{-iz}}\, dz \right| \le \int_{-\pi}^{\pi} \frac{|x| + |R|}{ae^R - 1}\, dx \le \int_{-\pi}^{\pi} \frac{\pi + |R|}{ae^R - 1}\, dx$$

Now $\lim_{R \to \infty} \dfrac{2\pi(\pi + |R|)}{ae^R - 1} = 0$, so $\int_{CD} f(z)\, dz = 0$.

2.

$$\int_{AB} f(z)\, dz = \int_{-\pi}^{\pi} \frac{x\, dx}{1 - ae^{-ix}} = \int_{-\pi}^{0} \frac{x\, dx}{1 - ae^{-ix}} + \int_{0}^{\pi} \frac{x\, dx}{1 - ae^{-ix}}$$

Changing x to $-x$ in the first integral, we get

$$\begin{aligned} \int_{AB} f(z)\, dz &= \int_0^{\pi} \frac{x\, dx}{1 - ae^{-ix}} - \int_0^{\pi} \frac{x\, dx}{1 - ae^{ix}} \\ &= \int_0^{\pi} \frac{ax(-e^{ix} + e^{-ix})}{1 - a(e^{ix} + e^{-ix}) + a^2}\, dx \\ &= \int_0^{\pi} \frac{-2iax \sin x}{1 - 2a \cos x + a^2}\, dx \end{aligned}$$

3. On BC, $z = \pi + iy$ and on DA, $z = -\pi + iy$, $dz = i\,dy$, and

$$\begin{aligned}
& \lim_{R\to\infty}\left[\int_{BC} f(z)\,dz + \int_{DA} f(z)\,dz\right] \\
= \;& i\int_0^\infty \frac{\pi + iy}{1 - ae^{y-i\pi}}\,dy + i\int_\infty^0 \frac{-\pi + iy}{1 - ae^{y+i\pi}}\,dy \\
= \;& i\int_0^\infty \left[\frac{\pi + iy}{1 + ae^y} - \frac{-\pi + iy}{1 + ae^y}\right] dy \\
= \;& 2\pi i\int_0^\infty \frac{e^{-y}}{e^{-y} + a}\,dy \;=\; -2\pi i \log(e^{-y} + a)\Big]_0^\infty \;=\; 2\pi i \log\frac{a+1}{a}
\end{aligned}$$

Thus

$$\lim_{R\to\infty}\int_C f(z)\,dz = \int_0^\pi \frac{-2iax\sin x}{1 - 2a\cos x + a^2}\,dx + 2\pi i\log\frac{a+1}{a} = \begin{cases} 0, & \text{when } a > 1 \\ -2\pi i\log a, & \text{when } 0 < a < 1 \end{cases}$$

showing that when $a > 1$,

$$\int_0^\pi \frac{x\sin x}{1 - 2a\cos x + a^2}\,dx = \frac{-2\pi i\log\frac{a+1}{a}}{-2ia} = \frac{\pi}{a}\log\left(1 + \frac{1}{a}\right)$$

and when $0 < a < 1$,

$$\int_0^\pi \frac{x\sin x}{1 - 2a\cos x + a^2}\,dx = \frac{-2\pi i\log a - 2\pi i\log\frac{a+1}{a}}{-2ia} = \frac{\pi}{a}\log(1 + a)$$

∎