UPSC Civil Services Main 1983 - Mathematics
Complex Analysis

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Question 1(a) *Obtain the Taylor and Laurent series expansions which represent the function* $\dfrac{z^2-1}{(z+2)(z+3)}$ *in the regions*
(i) $|z| < 2$ *(ii)* $2 < |z| < 3$ *(iii)* $|z| > 3$.

Solution. The only singularities of the function are at $z = -2$ and $z = -3$.

1. $|z| < 2$. In this region $f(z)$ is analytic and therefore will have Taylor series. It can be checked easily using partial fractions that

$$f(z) = 1 + \frac{3}{z+2} - \frac{8}{z+3}$$

Therefore

$$\begin{aligned} f(z) &= 1 + \frac{3}{2}\left(1+\frac{z}{2}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1} \\ &= 1 + \frac{3}{2}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{2}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n \\ &= -\frac{1}{6} + \sum_{n=1}^{\infty}(-1)^n\left[\frac{3}{2^{n+1}} - \frac{8}{3^{n+1}}\right]z^n \end{aligned}$$

is the required Taylor series valid in $|z| < 2$.

2. $2 < |z| < 3$: In this case we shall have a Laurent series.

$$\begin{aligned} f(z) &= 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{3}\left(1+\frac{z}{3}\right)^{-1} \\ &= 1 + \frac{3}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{2}{z}\right)^n - \frac{8}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n \\ &= 3\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{z^{n+1}} - \frac{5}{3} - \frac{8}{3}\sum_{n=1}^{\infty}(-1)^n\frac{z^n}{3^n} \end{aligned}$$

This is valid in $2 < |z| < 3$.

3. $|z| > 3$. We have a Taylor series around ∞ given by

$$f(z) = 1 + \frac{3}{z}\left(1+\frac{2}{z}\right)^{-1} - \frac{8}{z}\left(1+\frac{3}{z}\right)^{-1} = 1 + \sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}}(3\cdot 2^n - 8\cdot 3^n)$$

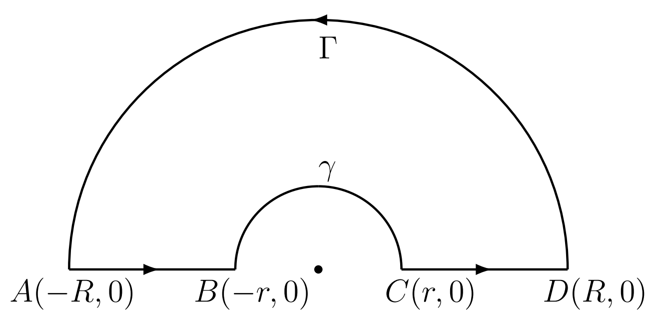
∎

Question 1(b) *Use the method of contour integration to evaluate*

$$\int_0^{\infty}\frac{x^{a-1}}{1+x^2}\,dx, \qquad 0 < a < 2$$

Solution.

We take $f(z) = \frac{z^{a-1}}{1+z^2}$ and the contour C as shown in the figure. We choose the branch of z^{a-1} which results in $f(x) = \frac{x^{a-1}}{1+x^2}$ on the real axis. The only pole of $f(z)$ inside C is at $z = i$. The residue at $z = i$ is $\lim_{z\to i}\dfrac{(z-i)z^{a-1}}{1+z^2} = \dfrac{i^{a-1}}{2i} = \dfrac{1}{2i}(e^{\frac{\pi i}{2}})^{a-1} = \dfrac{1}{2i}\left(\cos\dfrac{\pi(a-1)}{2} + i\sin\dfrac{\pi(a-1)}{2}\right)$.



Now

$$\left|\int_{\Gamma}\frac{z^{a-1}}{1+z^2}\,dz\right| \le \int_0^{\pi}\frac{R^{a-1}}{R^2-1}R\,d\theta \le \frac{\pi R^a}{R^2-1} \to 0 \text{ as } R\to\infty \because 0 < a < 2$$

and

$$\left|\int_{\gamma}\frac{z^{a-1}}{1+z^2}\,dz\right| \le \int_0^{\pi}\frac{r^{a-1}}{1-r^2}r\,d\theta \le \frac{\pi r^a}{r^2-1} \to 0 \text{ as } r\to 0 \because a > 0$$

Here we use $|1+z^2| \geq 1 - |z|^2 = 1 - r^2$. Thus

$$\begin{aligned}
\lim_{\substack{R\to\infty \\ r\to 0}} \int_C f(z)\,dz &= \int_{-\infty}^{0} f(xe^{i\pi})(-\,dx) + \int_0^\infty f(x)\,dx \\
&= \int_0^\infty \frac{x^{a-1}e^{i\pi(a-1)}}{1+x^2}\,dx + \int_0^\infty \frac{x^{a-1}}{1+x^2}\,dx \\
&= \int_0^\infty \frac{x^{a-1}}{1+x^2}\big(1+\cos\pi(a-1)+i\sin\pi(a-1)\big)\,dx \\
&= 2\pi i \cdot \frac{1}{2i}\Big(\cos\frac{\pi(a-1)}{2} + i\sin\frac{\pi(a-1)}{2}\Big)
\end{aligned}$$

Equating the real parts on both sides,

$$\big(1+\cos\pi(a-1)\big)\int_0^\infty \frac{x^{a-1}}{1+x^2}\,dx = \pi\cos\frac{\pi(a-1)}{2}$$

or

$$\int_0^\infty \frac{x^{a-1}}{1+x^2}\,dx = \pi\sec\frac{\pi(a-1)}{2}$$

Equating the imaginary parts also gives us the same answer. ∎

Alternate solution: In 1984, question 1(b), we obtained

$$\begin{aligned}
2\sin^2\frac{\pi a}{2}\int_0^\infty \frac{t^{a-1}\log t}{1+t^2}\,dt + \pi\sin\pi a\int_0^\infty \frac{t^{a-1}}{1+t^2}\,dt &= \frac{\pi^2}{2}\cos\frac{\pi a}{2} \\
-\sin\pi a\int_0^\infty \frac{t^{a-1}\log t}{1+t^2}\,dt - \pi\cos\pi a\int_0^\infty \frac{t^{a-1}}{1+t^2}\,dt &= \frac{\pi^2}{2}\sin\frac{\pi a}{2}
\end{aligned}$$

Multiplying the first by $\cos\frac{\pi a}{2}$ and the second by $\sin\frac{\pi a}{2}$ and adding gives us

$$\begin{aligned}
\big(\pi\sin\pi a\cos\frac{\pi a}{2} - \pi\cos\pi a\sin\frac{\pi a}{2}\big)\int_0^\infty \frac{t^{a-1}}{1+t^2}\,dt &= \frac{\pi^2}{2}\Big(\cos^2\frac{\pi a}{2} + \sin^2\frac{\pi a}{2}\Big) \\
\Longrightarrow \quad \pi\sin\Big(a\pi - \frac{a\pi}{2}\Big)\int_0^\infty \frac{t^{a-1}}{1+t^2}\,dt &= \frac{\pi^2}{2} \\
\Longrightarrow \quad \int_0^\infty \frac{t^{a-1}}{1+t^2}\,dt &= \frac{\pi}{2}\frac{1}{\sin\frac{a\pi}{2}} = \frac{\pi}{2\cos\big(\frac{\pi}{2}-\frac{a\pi}{2}\big)} = \frac{\pi}{2}\sec(a-1)\frac{\pi}{2}
\end{aligned}$$

as calculated before.

Note: In this solution the advantage is that we avoid the use of the multiple valued function $\log z$, however it is much longer.