UPSC Civil Services Main 1984 - Mathematics Complex Analysis

Brij Bhooshan

Asst. Professor

B.S.A. College of Engg & Technology

Mathura

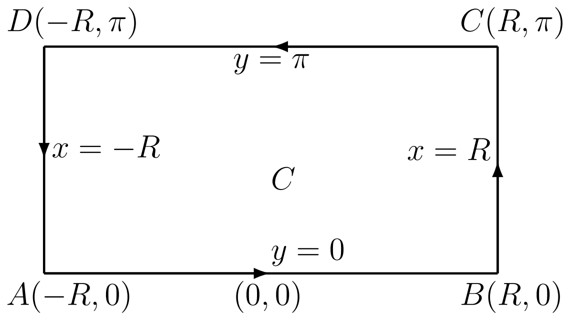
Question 1(a) *Evaluate by contour integration method* $\displaystyle\int_0^\infty \frac{x \sin mx}{x^4 + a^4}\, dx$.

Solution. See 1998 question 2(b). ∎

Question 1(b) *Evaluate by contour integration method* $\displaystyle\int_0^\infty \frac{x^{a-1} \log x}{1 + x^2}\, dx,\ 0 < a < 2$.

Solution.

We take $f(z) = \dfrac{ze^{az}}{1 + e^{2z}}$ and the contour C as the rectangle $ABCD$ where $A = (-R, 0), B = (R, 0), C = (R, \pi), D = (-R, \pi)$ oriented positively.



1. On $BC, z = R + iy$ and therefore

$$\left|\int_{BC} f(z)\, dz\right| = \left|\int_0^\pi \frac{(R + iy)e^{aR+iay}}{1 + e^{2(R+iy)}} i\, dy\right| \leq \int_0^\pi \frac{(R + y)e^{aR}}{e^{2R} - 1}\, dy \leq \frac{\pi(R + \pi)e^{aR}}{e^{2R} - 1}$$

because $|1 + e^{2R}| \geq e^{2R} - 1$ and $R + y \leq R + \pi$ on $0 \leq y \leq \pi$.

Since $\displaystyle\lim_{R\to\infty} \frac{(R + \pi)e^{aR}}{e^{2R} - 1} = \lim_{R\to\infty} \frac{(R + \pi)ae^{aR} + e^{aR}}{2e^{2R}} = \lim_{R\to\infty} \frac{(R + \pi)a + 1}{2e^{2R-aR}} = 0$ if $2-a > 0$ i.e. $a < 2$, it follows that $\displaystyle\lim_{R\to\infty} \int_{BC} f(z)\, dz = 0$.

2.
$$\left|\int_{DA} f(z)\,dz\right| = \left|\int_0^{\pi} \frac{(-R+iy)e^{-aR+iay}}{1+e^{2(-R+iy)}} i\,dy\right| \le \int_0^{\pi} \frac{(-R+y)e^{-aR}}{1-e^{-2R}}\,dy \le \frac{\pi(R+\pi)e^{-aR}}{1-e^{-2R}}$$

But $\lim_{R\to\infty} Re^{-aR} = 0$ (note that $e^{-aR} \le \frac{1}{a^2R^2}$), therefore $\lim\limits_{R\to\infty}\int_{DA} f(z)\,dz = 0$.

3. $\lim\limits_{R\to\infty}\int_{AB} f(z)\,dz = \int_{-\infty}^{\infty} \frac{xe^{ax}}{1+e^{2x}}\,dx.$

4. $\lim\limits_{R\to\infty}\int_{CD} f(z)\,dz = \int_{\infty}^{-\infty} \frac{(x+i\pi)e^{a(x+i\pi)}}{1+e^{2x+2i\pi}}\,dx$ as $z = x+i\pi$.

Thus

$$\begin{aligned}\lim_{R\to\infty}\int_C f(z)\,dz &= \int_{-\infty}^{\infty} \frac{xe^{ax}}{1+e^{2x}}\,dx - \int_{-\infty}^{\infty} \frac{(x+i\pi)e^{a(x+i\pi)}}{1+e^{2x}}\,dx \\ &= \int_{-\infty}^{\infty} \frac{xe^{ax}(1-e^{i\pi a})}{1+e^{2x}}\,dx - i\pi\int_{-\infty}^{\infty} \frac{e^{ax}e^{i\pi a}}{1+e^{2x}}\,dx\end{aligned}$$

The poles of $f(z)$ are given by $e^{2z} = e^{(2n+1)\pi i}$. The only pole in the strip $0 \le y \le \pi$ is $z = \frac{\pi i}{2}$ and it is a simple pole.

Residue at $z = \frac{\pi i}{2}$ is $\lim\limits_{z\to\frac{\pi i}{2}} \frac{ze^{az}(z-\frac{\pi i}{2})}{1+e^{2z}} = \frac{\frac{\pi i}{2}e^{\frac{\pi i a}{2}}}{2e^{\pi i}} = -\frac{\pi i}{4}e^{\frac{\pi i a}{2}}$. Thus

$$\int_{-\infty}^{\infty} \frac{xe^{ax}(1-e^{i\pi a})}{1+e^{2x}}\,dx - i\pi\int_{-\infty}^{\infty} \frac{e^{ax}e^{i\pi a}}{1+e^{2x}}\,dx = 2\pi i\left(-\frac{\pi i}{4}e^{\frac{\pi i a}{2}}\right) \quad (1)$$

Equating the real part of both sides, we get

$$\begin{aligned}\int_{-\infty}^{\infty} \frac{xe^{ax}(1-\cos\pi a)}{1+e^{2x}}\,dx + \pi\sin a\pi\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{2x}}\,dx &= \frac{\pi^2}{2}\cos\frac{\pi a}{2} \\ \Rightarrow 2\sin^2\frac{\pi a}{2}\int_{-\infty}^{\infty} \frac{xe^{ax}}{1+e^{2x}}\,dx + \pi\sin\pi a\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{2x}}\,dx &= \frac{\pi^2}{2}\cos\frac{\pi a}{2}\end{aligned}$$

Putting $e^x = t$ so that $x = \log t, dx = dt/t$, we get

$$2\sin^2\frac{\pi a}{2}\int_0^{\infty} \frac{t^{a-1}\log t}{1+t^2}\,dt + \pi\sin\pi a\int_0^{\infty} \frac{t^{a-1}}{1+t^2}\,dt = \frac{\pi^2}{2}\cos\frac{\pi a}{2} \quad (2)$$

Equating the imaginary parts in (1), we get

$$-\sin\pi a\int_{-\infty}^{\infty} \frac{xe^{ax}}{1+e^{2x}}\,dx - \pi\cos\pi a\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^{2x}}\,dx = \frac{\pi^2}{2}\sin\frac{\pi a}{2}$$

Put $e^x = t$ as before, to get

$$-\sin \pi a \int_0^\infty \frac{t^{a-1}\log t}{1+t^2}\,dt - \pi \cos \pi a \int_0^\infty \frac{t^{a-1}}{1+t^2}\,dt = \frac{\pi^2}{2}\sin\frac{\pi a}{2} \tag{3}$$

Multiplying (2) by $\cos \pi a$ and (3) by $\sin \pi a$ and adding we get

$$\begin{aligned}
\left(2\sin^2\frac{\pi a}{2}\cos\pi a - \sin^2\pi a\right)\int_0^\infty \frac{t^{a-1}\log t}{1+t^2}\,dt &= \frac{\pi^2}{2}\left[\cos\pi a\cos\frac{\pi a}{2} + \sin\pi a \sin\frac{\pi a}{2}\right] \\
&= \frac{\pi^2}{2}\cos\left(\pi a - \frac{\pi a}{2}\right) = \frac{\pi^2}{2}\cos\frac{\pi a}{2} \\
\text{Now}\quad 2\sin^2\frac{\pi a}{2}\cos\pi a - \sin^2\pi a &= 2\sin^2\frac{\pi a}{2}\left[\cos\pi a - 2\cos^2\frac{\pi a}{2}\right] \\
&= 2\sin^2\frac{\pi a}{2}\left[2\cos^2\frac{\pi a}{2} - 1 - 2\cos^2\frac{\pi a}{2}\right] \\
&= -2\sin^2\frac{\pi a}{2} \\
\Rightarrow \quad \int_0^\infty \frac{t^{a-1}\log t}{1+t^2}\,dt &= -\frac{\pi^2}{2}\cos\frac{\pi a}{2}\Big/ 2\sin^2\frac{\pi a}{2} \\
&= -\frac{\pi^2}{4}\cot\frac{\pi a}{2}\csc\frac{\pi a}{2}
\end{aligned}$$

as required. ∎

Question 1(c) *Distinguish clearly between a pole and an essential singularity. If $z = a$ is an essential singularity of a function $f(z)$, prove that for any positive numbers η, ρ, ϵ there exists a point z such that $0 < |z - a| < \rho$ for which $|f(z) - \eta| < \epsilon$.*

Solution. If $f(z)$ has an isolated singularity at z_0, which is not a removable singularity, then $f(z)$ has a pole at $z = z_0$ if $\lim_{z\to z_0} f(z) = \infty$. In this case if $f(z)$ has a pole of order k at $z = z_0$, then

$$f(z) = a_{-k}(z-z_0)^{-k} + \ldots + a_{-1}(z-z_0)^{-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

and this Laurent expansion is valid in some deleted neighborhood $0 < |z - z_0| < \delta$ of z_0.

If $\lim_{z\to z_0} f(z)$ does not exist, then $f(z)$ has an essential singularity at $z = z_0$. (Note that $\lim_{z\to z_0} f(z)$ is not finite as z_0 is not a removable singularity). In this case

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n$$

and $a_{-n} \neq 0$ for infinitely many n. Again this Laurent expansion is valid in some deleted neighborhood $0 < |z - z_0| < \delta$ of z_0.

The second part is Casorati-Weierstrass theorem. Let $f(z)$ be analytic in some deleted neighborhood N of a. Suppose that there exists $\epsilon > 0$ such that $|f(z)-\eta| < \epsilon$ is not satisfied for any $z \in N$i.e. $|f(z) - \eta| \geq \epsilon$ for every $z \in N$. Let $g(z) = \frac{1}{f(z)-\eta}$. Then $g(z)$ is analytic in N and $g(z)$ is bounded in N, therefore $g(z)$ has a removable singularity at a. Since $g(z)$ is not constant as $f(z)$ is not constant, either $g(a) \neq 0$ or $g(z)$ has a zero of order $k > 0$ at $z = a$. This means that $f(z) - \eta$ is either analytic at $z = a$ or $f(z) - \eta$ has a pole of order k at $z = a$. But this is not true, because $f(z)$ has an essential singularity at $z = a$. Thus our assumption is false i.e. we must have $z \in N$ for which $|f(z) - \eta| < \epsilon$. Note that we could take our deleted neighborhood N of the type $0 < |z - a| < \delta \leq \rho$. ∎