UPSC Civil Services Main 1985 - Mathematics
Complex Analysis

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Mathura

Question 1(a) *Prove that every power series represents an analytic function within its circle of convergence.*

Solution. Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ have R as its radius of convergence. We shall show that for any z in the region $C = \{z \ : \ |z| < R\}, f'(z) = \sum_{n=1}^{\infty} na_n z^{n-1}$. We first of all note that the radius of convergence of the series $\sum_{n=1}^{\infty} na_n z^{n-1}$ is also R as $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

Let $z \in C$ and $|z| < \rho < R$ and let h be chosen so small that $|z| + |h| \leq \rho < R$. Thus

$$\left|\frac{(z+h)^n - z^n}{(z+h) - z}\right| \leq (|z| + |h|)^{n-1} + |z|(|z| + |h|)^{n-2} + \ldots + |z|^{n-1} \leq n\rho^{n-1} \tag{1}$$

Since the series $\sum_{n=1}^{\infty} na_n \rho^{n-1}$ is convergent, given $\epsilon > 0$ $\exists N_1 > 0$ such that

$$\left|\sum_{r=n+1}^{\infty} r|a_r|\rho^{r-1}\right| < \frac{\epsilon}{3} \ \text{ for } n \geq N_1$$

and in particular $\sum_{r=N_1+1}^{\infty} r|a_r|\rho^{r-1} < \frac{\epsilon}{3}$. (2)

Since $\lim_{h\to 0}\left[a_n \frac{(z-h)^n - z^n}{h} - na_n z^{n-1}\right] = 0$, given $\epsilon > 0$ there exists $\delta > 0$ such that

$$\left|\sum_{n=1}^{N_1}\left[a_n \frac{(z-h)^n - z^n}{h} - na_n z^{n-1}\right]\right| < \frac{\epsilon}{3} \text{ for } |h| < \delta \tag{3}$$

Now

$$
\begin{aligned}
&\left|\frac{f(z+h)-f(z)}{h} - \sum_{n=1}^{\infty} n a_n z^{n-1}\right| \\
\leq\ & \left|\sum_{n=1}^{N_1}\left[a_n \frac{(z+h)^n - z^n}{h} - n a_n z^{n-1}\right]\right| + \sum_{n=N_1+1}^{\infty} \frac{|a_n\big((z+h)^n - z^n\big)|}{h} + \sum_{n=N_1+1}^{\infty} |n a_n z^{n-1}| \\
=\ & \frac{\epsilon}{3} + \sum_{n=N_1+1}^{\infty} |a_n| n \rho^{n-1} + \sum_{n=N_1+1}^{\infty} |a_n| n \rho^{n-1} \quad \text{for } |h| < \delta \\
<\ & \epsilon
\end{aligned}
$$

Thus $\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \sum_{n=1}^{\infty} n a_n z^{n-1} = f'(z)$, so $f(z)$ is analytic in C. ∎

Question 1(b) *Prove that the derivative of a function analytic in a domain is itself an analytic function.*

Solution. Cauchy's integral formula states that if $f(z)$ is analytic within and on a simple closed countour C oriented positively and if z_0 is any interior point of C, then $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0}$.

Let $f(z)$ be differentiable in a domain D and $z_0 \in D$. Let C be a circle with center z_0, the boundary of which is positively oriented, such that $f(z)$ is differentiable within and on C, and C along with its interior lies in D. Then by Cauchy's integral formula, $f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0}$.

Let $h \in \mathbb{C}$ be so small that $z_0 + h$ also lies in the interior of C.

$$
\begin{aligned}
\frac{f(z_0+h)-f(z_0)}{h} &= \frac{1}{2\pi i h}\int_C \left(\frac{f(z)}{z - z_0 - h} - \frac{f(z)}{z - z_0}\right) dz \\
&= \frac{1}{2\pi i h}\int_C \frac{h f(z)\,dz}{(z - z_0 - h)(z - z_0)} \\
&= \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z - z_0 - h)(z - z_0)}
\end{aligned}
$$

Now

$$
\begin{aligned}
&\frac{f(z_0+h)-f(z_0)}{h} - \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z - z_0)^2} \\
=\ & \frac{1}{2\pi i}\int_C \left(\frac{f(z)}{(z - z_0 - h)(z - z_0)} - \frac{f(z)}{(z - z_0)^2}\right) dz \\
=\ & \frac{1}{2\pi i}\int_C \frac{h f(z)\,dz}{(z - z_0 - h)(z - z_0)^2}
\end{aligned}
$$

Let $M = \sup_{z\in C} |f(z)|, l =$ length of $C, d = \min_{z\in C} |z - z_0|, d > 0$. Since we are interested in $h \to 0$, we could have assumed in the beginning itself that $0 < |h| < d$. Thus we get

$$\left| \frac{f(z_0 + h) - f(z_0)}{h} - \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^2} \right| \leq \frac{M|h|l}{2\pi d^2(d - |h|)}$$

Since the right hand side of the above inequality tends to 0 as $h \to 0$, it follows that

$$\lim_{h\to 0} \frac{f(z_0 + h) - f(z_0)}{h} = \frac{1}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^2}$$

i.e. $f(z)$ is differentiable at z_0 and since z_0 is an arbitrary point of D, it follows that

$$f'(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)^2}$$

where C is any positively oriented circle containing z in its interior.

We shall now prove that

$$f''(z_0) = \frac{2!}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^3}$$

where z_0, C are as chosen above. Let h be also chosen as above. Then

$$\begin{aligned}
& \frac{f'(z_0 + h) - f'(z_0)}{h} - \frac{2!}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^3} \\
= \ & \frac{1}{2\pi i h} \int_C f(z) \left[\frac{1}{(z - z_0 - h)^2} - \frac{1}{(z - z_0)^2} - \frac{2h}{(z - z_0)^3} \right] dz \\
= \ & \frac{1}{2\pi i h} \int_C f(z) \frac{(z - z_0)^3 - (z - z_0 - h)^2(z - z_0) - 2h(z - z_0 - h)^2}{(z - z_0 - h)^2 (z - z_0)^3}\, dz
\end{aligned}$$

$$\begin{aligned}
\text{Now} \quad & (z - z_0)^3 - (z - z_0 - h)^2(z - z_0) - 2h(z - z_0 - h)^2 \\
= \ & (z - z_0)[(z - z_0)^2 - (z - z_0 - h)^2] - 2h[(z - z_0)^2 - 2h(z - z_0) + h^2] \\
= \ & (z - z_0)h[2(z - z_0) - h] - 2h(z - z_0)^2 + 4h^2(z - z_0) - 2h^3 \\
= \ & h[2(z - z_0)^2 - h(z - z_0) - 2(z - z_0)^2 + 4h(z - z_0) - 2h^2] \\
= \ & h^2[3(z - z_0) - 2h]
\end{aligned}$$

Thus we get

$$\left| \frac{f'(z_0 + h) - f'(z_0)}{h} - \frac{2!}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^3} \right| \leq \frac{M|h|(3\rho + 2|h|^2)l}{2\pi d^3(d - |h|)^2}$$

where M, d, ρ are as before. Since the right hand side of the above inequality tends to 0 as $h \to 0$, it follows that

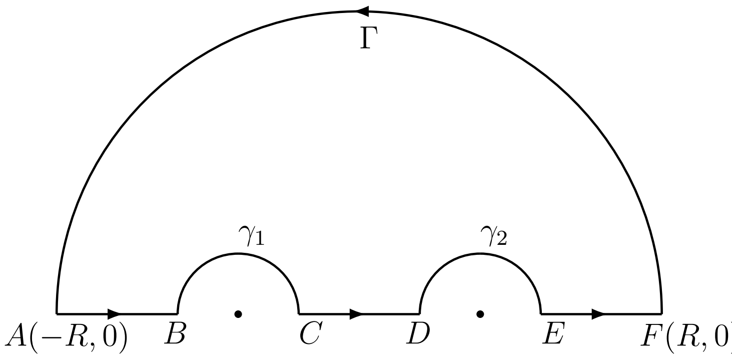
$$f''(z_0) = \lim_{h\to 0} \frac{f'(z_0 + h) - f'(z_0)}{h} = \frac{2!}{2\pi i} \int_C \frac{f(z)\, dz}{(z - z_0)^3}$$

i.e. $f'(z)$ is also analytic in D. ∎

Question 1(c) *Evaluate by the method of contour integration* $\displaystyle\int_0^\infty \frac{x \sin ax}{x^2 - b^2}\, dx$.

Solution. We take $f(z) = \frac{ze^{iaz}}{z^2-b^2}$ and the contour C consisting of the following

1. The line AB joining $A = (-R, 0)$ and $B = (-b - r_1, 0)$.
2. γ_1, the semicircle $(x+b)^2+y^2 = r_1^2$ lying in the upper half plane.
3. Line CD joining $C = (-b + r_1, 0)$ and $D = (b - r_2, 0)$.
4. γ_2, the semicircle $(x-b)^2+y^2 = r_2^2$ lying in the upper half plane.
5. Line EF joining $E = (b + r_2, 0)$ and $F = (R, 0)$.
6. Γ, the semicircle $x^2 + y^2 = R^2$ lying in the upper half plane.



Eventually we will let $R \to \infty, r_1, r_2 \to 0$. Now the integrand has no pole in the upper half plane, therefore

$$\lim_{\substack{R\to\infty \\ r_1\to 0 \\ r_2\to 0}} \int_C \frac{ze^{iaz}\, dz}{(z^2 - b^2)} = 0$$

1. On Γ,

$$\left|\int_\Gamma \frac{ze^{iaz}\, dz}{(z^2 - b^2)}\right| \leq \left|\int_0^\pi \frac{Re^i\theta e^{iaRe^{i\theta}}}{R^2 - b^2} Rie^{i\theta}\, d\theta\right|$$

because of Γ, $|z^2 - b^2| \geq |z|^2 - b^2 = R^2 - b^2$.

$$\left|\int_\Gamma \frac{ze^{iaz}\, dz}{(z^2 - b^2)}\right| \leq \frac{R^2}{R^2 - b^2}\int_0^\pi e^{-aR\sin\theta}\, d\theta = \frac{2R^2}{R^2 - b^2}\int_0^{\frac{\pi}{2}} e^{-aR\sin\theta}\, d\theta$$

(We can double the integral and halve the limit, because $\sin(\pi - \theta) = \sin\theta$). Using Jordan's inequality $\sin\theta \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$ we get

$$\left|\int_\Gamma \frac{ze^{iaz}\, dz}{(z^2 - b^2)}\right| \leq \frac{2R^2}{R^2 - b^2}\int_0^{\frac{\pi}{2}} e^{-aR\frac{2\theta}{\pi}}\, d\theta = \frac{2R^2}{R^2 - b^2}\left(\frac{1 - e^{-aR}}{2aR/\pi}\right) = \frac{\pi R(1 - e^{-aR})}{a(R^2 - b^2)}$$

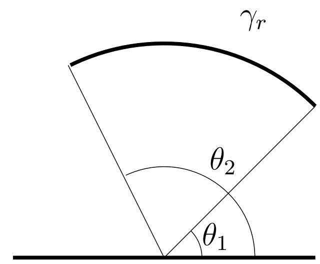
showing that $\displaystyle\lim_{R\to\infty}\int_\Gamma \frac{ze^{iaz}\, dz}{(z^2 - b^2)} = 0$.

2.

To get the value of the integral along γ_1, γ_2 we observe that if $f(z)$ has a simple pole at $z = a$ and γ_r is a part of a circle of radius r with center a, then

$$\lim_{r\to 0} \int_{\gamma_r} f(z)\, dz = ia_{-1}(\theta_2 - \theta_1)$$

where a_{-1} is the residue of $f(z)$ at a.



Proof: Let

$$f(z) = \frac{a_{-1}}{z-a} + a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots = \frac{a_{-1}}{z-a} + \phi(z)$$

where $\phi(z)$ is analytic in the circle $|z - a| \le r$. Thus

$$\left| \int_{\gamma_r} \phi(z)\, dz \right| \le Mr(\theta_2 - \theta_1)$$

where $M = \sup_{|z-a|=r} |\phi(z)|$. Thus $\lim_{r\to 0} \int_{\gamma_r} \phi(z)\, dz = 0$ and

$$\lim_{r\to 0} \int_{\gamma_r} f(z)\, dz = \int_{\gamma_r} \frac{a_{-1}\, dz}{z-a} = i \int_{\theta_1}^{\theta_2} a_{-1}\, d\theta = ia_{-1}(\theta_2 - \theta_1)$$

Now the residue of $\dfrac{ze^{iaz}}{z^2 - b^2}$ at $z = b$ is $\frac{1}{2}e^{iab}$, and the residue at $z = -b$ is $\frac{1}{2}e^{-iab}$. Thus $\lim_{r_1\to 0} \int_{\gamma_1} f(z)\, dz = \frac{1}{2}ie^{-iab}(0-\pi) = -\frac{i\pi}{2}e^{-iab}$ and $\lim_{r_2\to 0} \int_{\gamma_2} f(z)\, dz = \frac{1}{2}ie^{iab}(0-\pi) = -\frac{i\pi}{2}e^{iab}$.

Using the above data we get

$$0 = \lim_{\substack{R\to\infty \\ r_1\to 0 \\ r_2\to 0}} \int_C \frac{ze^{iaz}\, dz}{(z^2 - b^2)} = \int_{-\infty}^{\infty} \frac{xe^{iax}\, dx}{(x^2 - b^2)} - \frac{i\pi}{2}e^{-iab} - \frac{i\pi}{2}e^{iab}$$

or

$$\int_{-\infty}^{\infty} \frac{xe^{iax}\, dx}{(x^2 - b^2)} = \pi i \cos(ab)$$

Taking imaginary parts, we get

$$\int_{-\infty}^{\infty} \frac{x \sin ax\, dx}{(x^2 - b^2)} = \pi \cos(ab)$$

or

$$\int_0^{\infty} \frac{x \sin ax\, dx}{(x^2 - b^2)} = \frac{\pi \cos(ab)}{2}$$

∎