UPSC Civil Services Main 1986 - Mathematics
Complex Analysis

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Mathura

Question 1(a) *Let $f(z)$ be single valued and analytic within and on a simple closed curve C. If z_0 is any point in the interior of C, then show that*

$$f(z_0) = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0}$$

where the integral in taken in the positive sense around C.

Solution. This is known as the Cauchy integral formula. We shall show that given $\epsilon > 0$

$$\left|\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0} - f(z_0)\right| < \epsilon$$

which implies the result as ϵ is arbitrary.

Since $f(z)$ is analytic at z_0, it is continuous at z_0, therefore given $\epsilon > 0$ as above, there exists a $\delta > 0$ such that $|z - z_0| \leq \delta \Longrightarrow |f(z) - f(z_0)| < \epsilon$. We choose $\delta > 0$ so small that the disc $|z - z_0| < \delta$ lies within the interior of C. Then by Cauchy-Goursat's theorem (See 1987, 1(b)) we have

$$\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z - z_0} = \frac{1}{2\pi i}\int_\gamma \frac{f(z)\,dz}{z - z_0}$$

where γ is the circle $|z - z_0| = \rho < \delta$ and is positively oriented.

Now put $z - z_0 = \rho e^{i\theta}$ to get

$$\int_\gamma \frac{dz}{z - z_0} = \int_0^{2\pi} \frac{i\rho e^{i\theta}\,d\theta}{\rho e^{i\theta}} = 2\pi i$$

Therefore

$$\begin{aligned}\left|\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0} - f(z_0)\right| &= \left|\frac{1}{2\pi i}\left[\int_C \frac{f(z)\,dz}{z-z_0} - f(z_0)\int_\gamma \frac{dz}{z-z_0}\right]\right| \\ &= \left|\frac{1}{2\pi i}\int_\gamma \frac{f(z)-f(z_0)}{z-z_0}\,dz\right| \\ &\leq \frac{1}{2\pi\rho}\epsilon\int_\gamma |dz| = \frac{\epsilon}{2\pi\rho}\text{length of } \gamma = \epsilon\end{aligned}$$

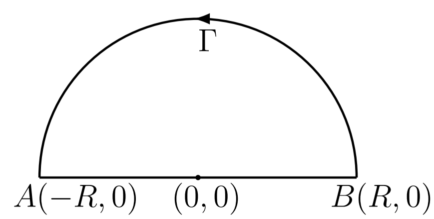
Thus $\frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{z-z_0} - f(z_0) = 0$ and the proof is complete. ∎

Question 1(b) *By the contour integration method show that*

1. $\int_0^\infty \frac{dx}{x^4+a^4} = \frac{\pi\sqrt{2}}{4a^2}$ *where* $a > 0$.
2. $\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$

Solution.

1. We take $f(z) = \frac{1}{z^4+a^4}$ and the contour C consisting of Γ a semicircle of radius R with center $(0,0)$ lying in the upper half plane, and the line AB joining $(-R,0)$ and $(R,0)$. C is positively oriented.



   (a) Poles of $f(z)$ are given by $z = \pm ae^{\pm\frac{\pi i}{4}} = \pm a[\cos\frac{\pi}{4} \pm i\sin\frac{\pi}{4}]$, out of which $z = ae^{\frac{\pi i}{4}}, z = -ae^{-\frac{\pi i}{4}}$ are in the upper half plane.

   Residue at $z = ae^{\frac{\pi i}{4}}$ is $\frac{1}{4a^3e^{\frac{3\pi i}{4}}} = \frac{1}{4a^3}\left[-\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right]^{-1}$.

   Residue at $z = -ae^{-\frac{\pi i}{4}}$ is $\frac{-1}{4a^3e^{-\frac{3\pi i}{4}}} = \frac{-1}{4a^3}\left[-\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right]^{-1}$.

   Sum of residues is $\frac{\sqrt{2}}{4a^3}\left[\frac{1}{i-1} + \frac{1}{1+i}\right] = -\frac{i\sqrt{2}}{4a^3}$. Thus

$$\lim_{R\to\infty}\int_C \frac{dz}{z^4+a^4} = \frac{2\pi i\cdot -i\sqrt{2}}{4a^3} = \frac{2\sqrt{2}\pi}{4a^3} = \frac{\pi}{\sqrt{2}a^3}$$

   (b)

$$\left|\int_\Gamma \frac{dz}{z^4+a^4}\right| \leq \left|\int_0^\pi \frac{Rie^{i\theta}}{R^4-a^4}\right| \leq \frac{\pi R}{R^4-a^4}$$

   because on Γ $|z^4+a^4| \geq |z^4| - a^4 = R^4 - a^4$. Thus $\lim_{R\to\infty}\int_\Gamma \frac{dz}{z^4+a^4} = 0$.

(c) $\lim_{R\to\infty}\int_{AB}\frac{dz}{z^4+a^4}=\int_{-\infty}^{\infty}\frac{dx}{x^4+a^4}=2\int_0^{\infty}\frac{dx}{x^4+a^4}$.

Using (a), (b) and (c) we get

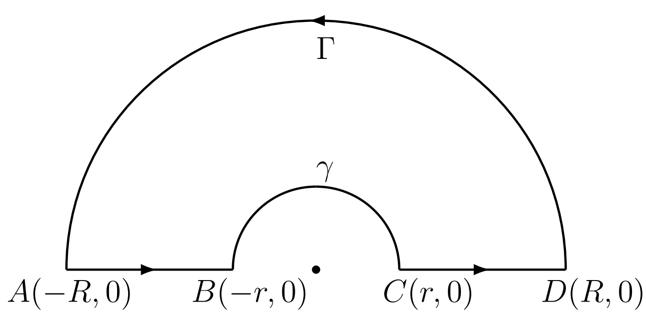
$$\lim_{R\to\infty}\int_C\frac{dz}{z^4+a^4}=2\int_0^{\infty}\frac{dx}{x^4+a^4}=\frac{\pi}{\sqrt{2}a^3}$$

Thus

$$\int_0^{\infty}\frac{dx}{x^4+a^4}=\frac{\pi}{2\sqrt{2}a^3}=\frac{\pi\sqrt{2}}{4a^3}$$

as required.

2. We take $f(z)=\frac{e^{iz}}{z}$ and the contour C consisting of the line AB joining $(-R,0)$ to $(-r,0)$, the semicircle γ of radius r with center $(0,0)$, the line CD joining $(r,0)$ to $(R,0)$ and Γ a semicircle of radius R with center $(0,0)$. The contour lies in the upper half plane and is oriented anticlockwise. We took γ as part of the contour to avoid the pole at $(0,0)$.

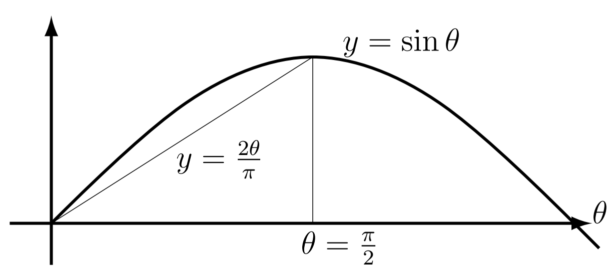


We will eventually make $R\to\infty$ and $r\to 0$.

(a) Since the integrand $\frac{e^{iz}}{z}$ has no poles in the upper half plane, it follows that

$$\lim_{R\to\infty, r\to 0}\int_C\frac{e^{iz}}{z}\,dz=0$$

(b) In order to prove that $\lim_{R\to\infty}\int_\Gamma\frac{e^{iz}}{z}\,dz=0$, we use Jordan inequality, which states that $\sin\theta\geq\frac{2\theta}{\pi}$ for $0\leq\theta\leq\frac{\pi}{2}$ — compare the graphs as shown in the figure.



$$\begin{aligned}\left|\int_\Gamma\frac{e^{iz}}{z}\,dz\right| &\leq \int_0^{\pi}\frac{e^{-R\sin\theta}}{R}R\,d\theta = 2\int_0^{\frac{\pi}{2}}e^{-R\sin\theta}\,d\theta \qquad (\because \sin(\pi-\theta)=\sin\theta)\\ &\leq 2\int_0^{\frac{\pi}{2}}e^{-R\frac{2\theta}{\pi}}\,d\theta = 2\cdot\frac{\pi}{2R}[1-e^{-R}]\end{aligned}$$

showing that $\lim_{R\to\infty}\int_\Gamma\frac{e^{iz}}{z}\,dz=0$ as $\lim_{R\to\infty}\frac{1-e^{-R}}{R}=0$.

(c) Now $\frac{e^{iz}}{z} = \frac{1}{z} + \phi(z)$ where $\phi(z)$ is analytic at $z = 0$. Thus given $\epsilon > 0, \exists \delta > 0$ such that $|z| < \delta \Rightarrow |\phi(z)| < \epsilon$. Thus $|\int_\gamma \phi(z)\,dz| \leq \epsilon \cdot (\text{length of } \gamma) \Rightarrow \lim_{r\to 0} \int_\gamma \phi(z)\,dz = 0$.

$$\int_\gamma \frac{dz}{z} = \int_\pi^0 \frac{re^{i\theta} i\,d\theta}{re^{i\theta}} = -i\pi$$

(d)

$$\lim_{R\to\infty, r\to 0} \int_{AB} \frac{e^{iz}}{z}\,dz = \int_{-\infty}^0 \frac{e^{ix}}{x}\,dx, \quad \lim_{R\to\infty, r\to 0} \int_{CD} \frac{e^{iz}}{z}\,dz = \int_0^\infty \frac{e^{ix}}{x}\,dx$$

Using these results, we get

$$0 = \lim_{R\to\infty, r\to 0} \int_C \frac{e^{iz}}{z}\,dz = \int_{-\infty}^0 \frac{e^{ix}}{x}\,dx - i\pi + \int_0^\infty \frac{e^{ix}}{x}\,dx \qquad (*)$$

Since

$$\int_{-\infty}^0 \frac{e^{ix}}{x}\,dx = -\int_0^\infty \frac{e^{-iy}}{y}\,dy$$

we get

$$\int_0^\infty \frac{e^{ix}}{x}\,dx - \int_0^\infty \frac{e^{-ix}}{x}\,dx = i\pi$$

or

$$\int_0^\infty \frac{e^{ix} - e^{-ix}}{2i} \frac{1}{x}\,dx = \frac{\pi}{2} \Rightarrow \int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2}$$

Note that in (*) we cannot write $\displaystyle\int_{-\infty}^\infty \frac{e^{ix}}{x}\,dx = \pi i$ and conclude that $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{x}\,dx = \pi, \int_{-\infty}^\infty \frac{\cos x}{x}\,dx = 0$, because $\displaystyle\int_{-\infty}^\infty \frac{\cos x}{x}\,dx$ has convergence problem at $x = 0$.

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