UPSC Civil Services Main 1987 - Mathematics Complex Analysis

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Mathura

Question 1(a) *By considering the Laurent series for* $f(z) = \dfrac{1}{(1-z)(z-2)}$ *prove that if* C *is a closed contour oriented in the counter clockwise direction, then* $\displaystyle\int_C f(z)\,dz = 2\pi i$.

Solution. Laurent's theorem states that if $f(z)$ is analytic throughout the annular region $R_1 < |z - z_0| < R_2$ and C is any positively oriented simple closed curve lying in the annular region and having z_0 in its interior, then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n \quad \text{where} \quad a_n = \frac{1}{2\pi i}\int_C \frac{f(z)\,dz}{(z-z_0)^{n+1}}$$

Thus $\displaystyle\int_C f(z)\,dz = 2\pi i a_{-1}$.

Now in our case: $R_1 = 1, R_2 = 2, z_0 = 0$ i.e. $f(z)$ is analytic in $1 < |z| < 2$. Moreover

$$f(z) = \frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1} + \frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1}$$

Since $|\frac{1}{z}| < 1$ and $|\frac{z}{2}| < 1$, we get $f(z) = \displaystyle\sum_{n=0}^{\infty} \frac{1}{z^{n+1}} + \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ as the Laurent expansion of $f(z)$ valid in $1 < |z| < 2$. Thus if C is a simple closed contour lying in $1 < |z| < 2$ with the origin in its interior, then $\displaystyle\int_C f(z)\,dz = 2\pi i$, since $a_{-1} = 1$.

Note that in this question, the curve C has not been clearly specified. If C is in the region $1 < |z| < 2$, but does not contain the origin, then $\displaystyle\int_C f(z)\,dz = 0$.

Note: What about the other cases — C lies in $|z| < 1$ or $|z| > 2$ ∎

Question 1(b) *State and prove Cauchy's residue theorem.*

Solution. Statement: If C is a simple closed contour oriented anticlockwise and $f(z)$ is a complex valued function which is analytic on and within the interior of C except for a finite number of poles $z_1, \ldots z_n$ in the interior of C, then

$$\int_C f(z)\,dz = 2\pi i \sum_{r=1}^{n} \text{Residue of } f(z) \text{ at } z = z_r$$

Proof: We enclose each $z_j, 1 \leq j \leq n$ is a small disc C_j such that C_j along with its boundary lies within C and, and these discs are small enough that C_j and C_k, $j \neq k$ do not overlap, or even touch on their boundaries. We now use Cauchy-Goursat theorem:

Theorem (Caucy-Goursat): If C is a simple closed positively oriented contour and $C_1, \ldots C_n$ are simple closed positively oriented contours which lie within C, and whose interiors have no points in common, and if $f(z)$ is a function which is analytic within and on C except for the interiors of C_j, $1 \leq j \leq n$, then

$$\int_C f(z)\,dz = \sum_{j=1}^{n} \int_{C_j} f(z)\,dz$$

Using this theorem we see that in our case

$$\int_C f(z)\,dz = \sum_{j=1}^{n} \int_{C_j} f(z)\,dz$$

But in the last question we have seen that

$$\int_{C_j} f(z)\,dz = 2\pi i \times \text{Residue of } f(z) \text{ at } z = z_j$$

(This is Laurent's theorem.) Thus

$$\int_C f(z)\,dz = 2\pi i \sum_{r=1}^{n} \text{Residue of } f(z) \text{ at } z = z_r$$

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Question 1(c) *By the method of contour integration show that*

$$\int_0^{\infty} \frac{\cos x}{x^2 + a^2}\,dx = \frac{\pi e^{-a}}{2a}, \;\; a > 0$$

Solution. See 2002, question 2(b). ∎