UPSC Civil Services Main 1988 - Mathematics Complex Analysis

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Mathura

Question 1(a) *By evaluating* $\int \frac{dz}{z+2}$ *over a suitable contour* C *prove that*

$$\int_0^{\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta = 0$$

Solution. By using the unit circle $|z| = 1$ as contour, and integrating $\int_{|z|=1} \frac{dz}{z+2}$, we have proved $\int_0^{2\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta = 0$ — see 1997, question 1(b). Now in $\int_{\pi}^{2\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta$ put $\theta = 2\pi - \phi$ so that

$$\int_{\pi}^{2\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta = \int_{\pi}^{0} \frac{1 + 2\cos(2\pi - \phi)}{5 + 4\cos(2\pi - \phi)}\,(-d\phi) = \int_0^{\pi} \frac{1 + 2\cos\phi}{5 + 4\cos\phi}\, d\phi$$

Thus $\int_0^{2\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta = 2\int_0^{\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta$ showing that

$$\int_0^{\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta = 0$$

. ∎

Note: If the contour was not prescribed, we could have put $z = e^{i\theta}$ to get

$$\int_0^{2\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta = \frac{1}{i}\int_{|z|=1} \frac{z^2 + z + 1}{z(5z + 2z^2 + 2)}\, dz$$

The integrand has two poles at $z = 0, z = -\frac{1}{2}$ inside $|z| = 1$, which are simple poles. The residue at $z = 0$ is $\frac{1}{2}$ and the residue at $z = -\frac{1}{2}$ is $-\frac{1}{2}$, so we get

$$\int_{|z|=1} \frac{z^2 + z + 1}{z(5z + 2z^2 + 2)}\, dz = 0 \Rightarrow \int_0^{2\pi} \frac{1 + 2\cos\theta}{5 + 4\cos\theta}\, d\theta = 0$$

Question 1(b) *If $f(z)$ is analytic in $|z| \leq R$ and x, y lie inside the disc, evaluate the integral $\displaystyle\int_{|z|=R} \frac{f(z)\, dz}{(z - x)(z - y)}$ and deduce that a function analytic and bounded for all finite z is a constant.*

Solution. Cauchy's integral formula states that if $f(z)$ is analytic on and within the disc $|z| \leq R$, then for any ζ which lies within the disc

$$f(\zeta) = \frac{1}{2\pi i} \int_{|z|=R} \frac{f(z)\, dz}{\zeta - z}$$

Thus

$$\int_{|z|=R} \frac{f(z)\, dz}{(z - x)(z - y)} = \frac{1}{x - y}\left[\int_{|z|=R} \frac{f(z)\, dz}{z - x} - \int_{|z|=R} \frac{f(z)\, dz}{z - y}\right] = \frac{2\pi i}{x - y}\left[f(x) - f(y)\right]$$

We now prove the remaining part, which is Liouville's theorem.

Let $|f(z)| \leq M$ for every z. Clearly $|z - x| \geq |z| - |x| = R - |x|$ and similarly $|z - y| \geq R - |y|$ on $|z| = R$, and therefore

$$\left|\int_{|z|=R} \frac{f(z)\, dz}{(z - x)(z - y)}\right| \leq \frac{M \cdot 2\pi R}{(R - |x|)(R - |y|)}$$

Thus $|f(x) - f(y)| \leq \left|\frac{1}{2\pi i}\right| \frac{|x - y| \cdot M \cdot 2\pi R}{(R - |x|)(R - |y|)}$. Since $\frac{R}{(R - |x|)(R - |y|)} \to 0$ as $R \to \infty$, it follows that $|f(x) - f(y)| = 0$ or $f(x) = f(y)$, so f is constant. ∎

Question 1(c) *If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence R and $0 < r < R$, prove that*

$$\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

Solution.

$$|f(z)|^2 = f(z) \cdot \overline{f(z)} = \sum_{n=0}^{\infty} a_n z^n \sum_{m=0}^{\infty} \overline{a_m}\, \overline{z}^m = \sum_{n=0}^{\infty} \sum_{p+q=n} a_p \overline{a_q} z^p \overline{z}^q$$

We know that if a power series has a radius of convergence R, then it is uniformly and absolutely convergent in $|z| \leq r$ where $0 < r < R$, therefore

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z)|^2 \, d\theta = \sum_{n=0}^{\infty} \frac{1}{2\pi}\int_0^{2\pi} \sum_{p+q=n} a_p \overline{a_q} r^p r^q e^{i(p-q)\theta} \, d\theta$$

Since $\int_0^{2\pi} e^{i(p-q)\theta} \, d\theta = 0$ when $p \neq q$, we get

$$\frac{1}{2\pi}\int_0^{2\pi} |f(z)|^2 \, d\theta = \sum_{n=0}^{\infty} |a_n|^2 r^{2n}$$

(Note: This shows that if $|f(z)| \leq M$ on $|z| = r$, then $\sum_{n=0}^{\infty} |a_n|^2 r^{2n} \leq M^2$.) ∎

Question 2(a) *Evaluate* $\int_C \frac{ze^z \, dz}{(z-a)^3}$ *if* a *lies inside the closed contour* C*.*

Solution. Clearly the only pole of $\frac{ze^z}{(z-a)^3}$ is of order 3 at $z = a$. The residue at this pole is

$$\frac{1}{2!}\frac{d^2}{dz^2}\left(\frac{(z-a)^3 ze^z}{(z-a)^3}\right)_{z=a} = \frac{1}{2}\frac{d}{dz}\left(ze^z + e^z\right)_{z=a} = \frac{1}{2}\left(ze^z + e^z + e^z\right)_{z=a} = e^a\left(1 + \frac{a}{2}\right)$$

Thus by Cauchy's residue theorem,

$$\int_C \frac{ze^z \, dz}{(z-a)^3} = 2\pi i \cdot e^a\left(1 + \frac{a}{2}\right) = \pi i e^a(2+a)$$

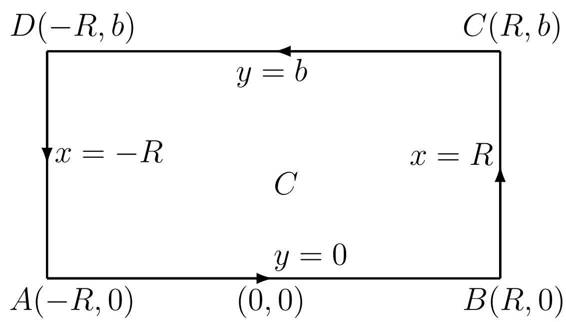
∎

Question 2(b) *Prove*

$$\int_0^{\infty} e^{-x^2} \cos(2bx) \, dx = \frac{\sqrt{\pi}}{2} e^{-b^2} \qquad (b > 0)$$

by integrating e^{-z^2} *along the boundary of the rectangle* $|x| \leq R, 0 \leq y \leq b$*.*

Solution.



Let the rectangle be $ABCD$ where $A = (-R, 0), B = (R, 0), C = (R, b), D = (-R, b)$ oriented positively. Since e^{-z^2} has no pole inside $ABCD$, we get $\lim_{R\to\infty} \int_{ABCD} e^{-z^2} \, dz = 0$.

(Note that e^{-z^2} has no pole in the entire complex plane.)

1. On BC, $z = R + iy$ and $0 \le y \le b$, therefore

$$\left|\int_{BC} e^{-z^2}\,dz\right| = \left|\int_0^b e^{-R^2} e^{-2Riy} e^{-i^2y^2}\,i\,dy\right| \le e^{-R^2}\int_0^b e^{y^2}\,dy = (\text{constant})e^{-R^2}$$

   Clearly $e^{-R^2} \to 0$ as $R \to \infty$, so $\lim_{R\to\infty}\int_{BC} e^{-z^2}\,dz = 0$.

2. On DA, $z = -R + iy$ and $0 \le y \le b$, therefore

$$\left|\int_{DA} e^{-z^2}\,dz\right| = \left|\int_b^0 e^{-R^2} e^{2Riy} e^{-i^2y^2}\,i\,dy\right| \le e^{-R^2}\int_0^b e^{y^2}\,dy$$

   Thus $\lim_{R\to\infty}\int_{DA} e^{-z^2}\,dz = 0$.

3. On AB, $z = x$ so $\lim_{R\to\infty}\int_{AB} e^{-z^2}\,dz = \int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

4. On CD, $z = x + ib$, therefore

$$\lim_{R\to\infty}\int_{CD} e^{-z^2}\,dz = \int_{\infty}^{-\infty} e^{-x^2} e^{-i^2b^2} e^{-2ibx}\,dx = -e^{b^2}\int_{-\infty}^{\infty} e^{-x^2}[\cos 2bx - i\sin 2bx]\,dx$$

Using the above calculations, we get

$$0 = \lim_{R\to\infty}\int_C e^{-z^2}\,dz = \sqrt{\pi} - e^{b^2}\int_{-\infty}^{\infty} e^{-x^2}[\cos 2bx - i\sin 2bx]\,dx$$

Equating real and imaginary parts,

$$\int_{-\infty}^{\infty} e^{-x^2}\sin 2bx\,dx = 0$$

and

$$\int_{-\infty}^{\infty} e^{-x^2}\cos 2bx\,dx = \sqrt{\pi}e^{-b^2}$$

Thus

$$\int_0^{\infty} e^{-x^2}\cos 2bx\,dx = \frac{1}{2}\int_{-\infty}^{\infty} e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}e^{-b^2}}{2}$$

∎

Question 2(c) *Prove that the coefficients c_n of the expansion*

$$\frac{1}{1-z-z^2} = \sum_{n=0}^{\infty} c_n z^n$$

satisfy $c_n = c_{n-1} + c_{n-2}, n \geq 2$. Determine c_n.

Solution. $z^2 + z - 1 = 0 \Rightarrow z = \frac{-1\pm\sqrt{5}}{2}$. Let $\lambda = \frac{-1+\sqrt{5}}{2}, \mu = \frac{-1-\sqrt{5}}{2}$. Thus $f(z) = \frac{1}{1-z-z^2}$ is analytic in the disc $|z| < \lambda$ as both the singularities at $z = \lambda$ and $z = \mu$ lie outside it. Thus $f(z)$ has Taylor series expansion with center $z = 0$.

Let $f(z) = \sum_{n=0}^{\infty} c_n z^n$, then $(1 - z - z^2)\sum_{n=0}^{\infty} c_n z^n = 1$. Equating coefficients of like powers we get

$$\begin{aligned} c_0 &= 1 \\ c_1 - c_0 &= 0 \\ c_2 - c_1 - c_0 &= 0 \\ &\dots \\ c_n - c_{n-1} - c_{n-2} &= 0 \end{aligned}$$

Thus $c_n = c_{n-1} + c_{n-2}, n \geq 2$. The c_n's are Fibonacci numbers.

Now

$$\begin{aligned} f(z) &= \frac{-1}{(z-\lambda)(z-\mu)} \\ &= \frac{-1}{\lambda-\mu}\left[\frac{1}{z-\lambda} - \frac{1}{z-\mu}\right] \\ &= \frac{-1}{\sqrt{5}}\left[-\frac{1}{\lambda}\left(1-\frac{z}{\lambda}\right)^{-1} - \frac{-1}{\mu}\left(1-\frac{z}{\mu}\right)^{-1}\right] \end{aligned}$$

If we confine z to the disc $|z| < \lambda$, then $|\frac{z}{\lambda}| < 1, |\frac{z}{\mu}| < 1$ and we have

$$f(z) = \frac{1}{\sqrt{5}}\left[\sum_{n=0}^{\infty}\frac{z^n}{\lambda^{n+1}} - \sum_{n=0}^{\infty}\frac{z^n}{\mu^{n+1}}\right] = \sum_{n=0}^{\infty} c_n z^n$$

where c_n are given as above. But the Taylor series of a function is unique, therefore we have

$$\begin{aligned} c_n &= \frac{1}{\sqrt{5}}\left[\frac{1}{\lambda^{n+1}} - \frac{1}{\mu^{n+1}}\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{2}{\sqrt{5}-1}\right)^{n+1} - \left(\frac{-2}{\sqrt{5}+1}\right)^{n+1}\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{2(\sqrt{5}+1)}{5-1}\right)^{n+1} - \left(\frac{-2(\sqrt{5}-1)}{5-1}\right)^{n+1}\right] \\ &= \frac{1}{\sqrt{5}}\left[\left(\frac{\sqrt{5}+1}{2}\right)^{n+1} + (-1)^n\left(\frac{\sqrt{5}-1}{2}\right)^{n+1}\right] \end{aligned}$$

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