UPSC Civil Services Main 1989 - Mathematics Complex Analysis

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Mathura

Question 1(a) *Evaluate the integral*

$$\frac{1}{2\pi i}\int_C \frac{e^z\,dz}{z(1-z)^3}$$

if

1. *the point* 0 *lies inside* C *and the point* 1 *lies outside* C.
2. *both* 0 *and* 1 *lie inside* C.
3. *the point* 1 *lies inside* C *and the point* 0 *lies outside* C.

Solution. The only possible poles of $\dfrac{e^z}{z(z-1)^3}$ are $z=0$ and $z=1$. Clearly $z=0$ is a simple pole, and residue at $z=0$ is $\lim\limits_{z\to 0}\dfrac{ze^z}{z(1-z)^3}=1$.

We have a triple pole at $z=1$, and the residue at $z=1$ is $\dfrac{1}{2!}\dfrac{d^2}{dz^2}\left(\dfrac{(z-1)^3e^z}{z(1-z)^3}\right)_{z=1} = -\dfrac{1}{2}\dfrac{d}{dz}\left(\dfrac{ze^z-e^z}{z^2}\right)_{z=1} = -\dfrac{1}{2}\left(\dfrac{(z^2(ze^z+e^z-e^z)-2z(ze^z-e^z)}{z^4}\right)_{z=1} = -\dfrac{e}{2}$.

By Cauchy's residue theorem, $\dfrac{1}{2\pi i}\displaystyle\int_C \dfrac{e^z\,dz}{z(1-z)^3}$ = Sum of the residues at poles of integrand within C.

1. The only pole inside C is 0, so

$$\frac{1}{2\pi i}\int_C \frac{e^z\,dz}{z(1-z)^3} = \text{Residue at } 0 = 1$$

2. Both poles are in C, so

$$\frac{1}{2\pi i}\int_C \frac{e^z\,dz}{z(1-z)^3} = \text{Residue at } 0 + \text{Residue at } 1 = 1 - \frac{e}{2}$$

3. The only pole inside C is 1, so

$$\frac{1}{2\pi i}\int_C \frac{e^z\,dz}{z(1-z)^3} = \text{Residue at } 1 = -\frac{e}{2}$$

∎

Question 1(b) *Let f have the Taylor expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$ in $|z| < R$ and let $s_n(z) = \sum_{k=0}^{m} a_k z^k$. If $0 < r < R$ and if $|z| < r$ show that*

$$s_n(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}\,\frac{w^{n+1}-z^{n+1}}{w-z}\,dw$$

where γ is the circle $|w| = r$ oriented positively.

Solution. Since $\dfrac{w^{n+1}-z^{n+1}}{w-z} = w^n + w^{n-1}z + \ldots + wz^{n-1} + z^n = \sum_{k=0}^{n} w^{n-k}z^k$, it follows that

$$\begin{aligned}
\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}\,\frac{w^{n+1}-z^{n+1}}{w-z}\,dw &= \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}\left(\sum_{k=0}^{n} w^{n-k}z^k\right)dw \\
&= \frac{1}{2\pi i}\int_\gamma \left(\sum_{k=0}^{n} z^k \frac{f(w)}{w^{k+1}}\right)dw \\
&= \frac{1}{2\pi i}\sum_{k=0}^{n} z^k \int_\gamma \frac{f(w)}{w^{k+1}}\,dw
\end{aligned}$$

But from Cauchy's integral formula we know that

$$\frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{k+1}}\,dw$$

Therefore

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}}\,\frac{w^{n+1}-z^{n+1}}{w-z}\,dw = \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} z^k$$

Since $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and the series is uniformly and absolutely convergent within $|z| \leq r$, we can differentiate it termwise. Thus we obtain $f(0) = a_0, f'(0) = a_1, f''(0) = 2a_2, \ldots, f^{(k)}(0) = k!a_k, \ldots$. Substituting above, we get

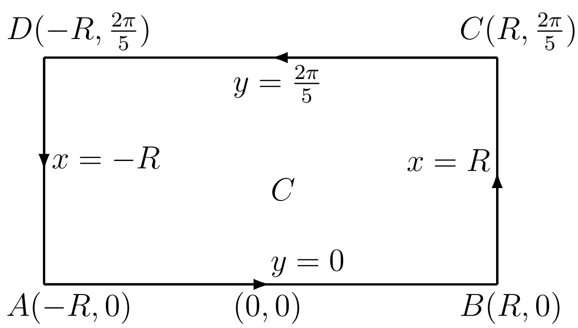
$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w^{n+1}} \frac{w^{n+1} - z^{n+1}}{w - z}\, dw = \sum_{k=0}^{n} a_k z^k = s_n(z)$$

∎

Question 1(c) *By integrating* $\dfrac{1}{1+z^5}$ *around a suitable contour, prove that*

$$\int_0^\infty \frac{dx}{1+x^5} = \frac{\pi}{5}\Big/\sin\frac{\pi}{5}$$

Solution. We shall present two proofs.



Proof 1: Let $f(z) = \frac{e^z}{1+e^{5z}}$ and the contour be C, the rectangle $ABCD$ where $A = (-R, 0), B = (R, 0), C = (R, \frac{2\pi}{5}), D = (-R, \frac{2\pi}{5})$ oriented positively. We let $R \to \infty$ eventually. The only pole in the strip bounded by $y = 0$ and $y = \frac{2\pi}{5}$ is $z = \frac{\pi i}{5}$ and it is a simple pole.

Residue of $f(z)$ at $z = \frac{\pi i}{5}$ is $\displaystyle\lim_{z \to \frac{\pi i}{5}} \frac{(z - \frac{\pi i}{5})e^z}{1 + e^{5z}} = \frac{e^{\frac{\pi i}{5}}}{5e^{i\pi}} = -\frac{e^{\frac{\pi i}{5}}}{5}$. Thus

$$\lim_{R\to\infty}\int_C \frac{e^z\, dz}{1+e^{5z}} = -\frac{2\pi i e^{\frac{\pi i}{5}}}{5}$$

Now we evaluate the integral on all 4 sides of the rectangle.

1. $$\left|\int_{BC} \frac{e^z\, dz}{1+e^{5z}}\right| = \left|\int_0^{\frac{2\pi}{5}} \frac{e^{R+iy}}{e^{5z}+1} i\, dy\right| \leq \int_0^{\frac{2\pi}{5}} \frac{e^R}{e^{5R}-1}\, dy \leq \frac{2\pi}{5}\,\frac{e^R}{e^{5R}-1}$$
   because $|e^{5z} + 1| \geq |e^{5z}| - 1 = |e^{5R+5iy}| - 1 = e^{5R} - 1$. as on BC, $z = R + iy$. Thus $\displaystyle\lim_{R\to\infty}\int_{BC} \frac{e^z\, dz}{1+e^{5z}} = 0$.

2. On DA, $z = -R + iy$ and therefore $|e^{5z} + 1| \geq 1 - |e^{5z}| = 1 - e^{-5R}$. This shows that
   $$\left|\int_{DA} \frac{e^z\, dz}{1+e^{5z}}\right| \leq \frac{2\pi}{5}\,\frac{e^{-R}}{1-e^{-5R}}$$
   As $\dfrac{e^{-R}}{1-e^{-5R}} \to 0$ as $R \to \infty$, it follows that $\displaystyle\lim_{R\to\infty}\int_{DA} \frac{e^z\, dz}{1+e^{5z}} = 0$.

3. On AB, $z = x$ so

$$\lim_{R\to\infty}\int_{AB}\frac{e^z\,dz}{1+e^{5z}} = \int_{-\infty}^{\infty}\frac{e^x\,dx}{1+e^{5x}}$$

4. On CD, $z = x + \frac{2\pi i}{5}$, so

$$\lim_{R\to\infty}\int_{CD}\frac{e^z\,dz}{1+e^{5z}} = \int_{\infty}^{-\infty}\frac{e^x e^{\frac{2\pi i}{5}}\,dx}{1+e^{5x}}$$

Using the above, we get

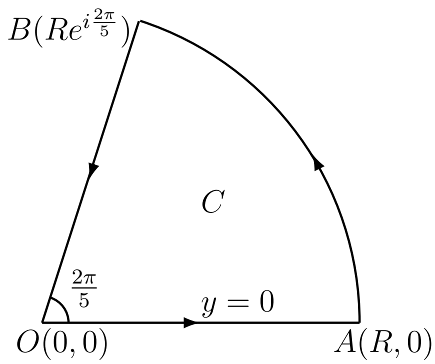
$$\lim_{R\to\infty}\int_{C}\frac{e^z\,dz}{1+e^{5z}} = \int_{-\infty}^{\infty}\frac{e^x\,dx}{1+e^{5x}} - e^{\frac{2\pi i}{5}}\int_{-\infty}^{\infty}\frac{e^x\,dx}{1+e^{5x}} = -\frac{2\pi i e^{\frac{\pi i}{5}}}{5}$$

or

$$\int_{-\infty}^{\infty}\frac{e^x\,dx}{1+e^{5x}} = -\frac{2\pi i}{5}\frac{e^{\frac{\pi i}{5}}}{1-e^{\frac{2\pi i}{5}}} = \frac{\pi}{5}\frac{2i}{e^{\frac{\pi i}{5}}-e^{\frac{\pi i}{5}}} = \frac{\pi}{5}\Big/\sin\frac{\pi}{5}$$

We now put $e^x = t$ to get $\displaystyle\int_0^{\infty}\frac{dt}{1+t^5} = \frac{\pi}{5}\Big/\sin\frac{\pi}{5}$ as desired.

Proof 2: Let $f(z) = \frac{1}{1+z^5}$ and the contour be C, the angular region $OABO$ where OA is the line joining $(0,0),(R,0)$, AB is the arc of the circle $|z| = R$ and B is on the circle such that angle $\angle AOB = \frac{2\pi}{5}$. C is oriented positively. We let $R\to\infty$ eventually. The only pole in the sector is $z = \frac{\pi i}{5}$ and it is a simple pole.



Using Cauchy's residue theorem, we get

$$\lim_{R\to\infty}\int_{C}\frac{dz}{1+z^5} = 2\pi i\times\text{Residue at } e^{\frac{\pi i}{5}} = 2\pi i\lim_{z\to e^{\frac{\pi i}{5}}}\frac{z-e^{\frac{\pi i}{5}}}{1+z^5} = \frac{2\pi i}{5e^{\frac{4\pi i}{5}}}$$

1. On AB, $z = Re^{i\theta}, |z^5+1| \geq |z|^5 - 1 = R^5 - 1$, $0\leq\theta\leq\frac{2\pi}{5}$ and therefore

$$\left|\int_{AB}\frac{dz}{1+z^5}\right| \leq \left|\int_0^{\frac{2\pi}{5}}\frac{Rie^{i\theta}\,d\theta}{R^5-1}\right| \leq \frac{2\pi}{5}\frac{R}{R^5-1}$$

   showing that $\displaystyle\lim_{R\to\infty}\int_{AB}\frac{dz}{1+z^5} = 0$.

2. On OA, $z = x$ and therefore $\displaystyle\lim_{R\to\infty}\int_{OA}\frac{dz}{1+z^5} = \int_0^{\infty}\frac{dx}{1+x^5}$.

3. On BO, $z = Re^{\frac{2\pi i}{5}}$ and R varies from ∞ to 0. Therefore

$$\lim_{R\to\infty}\int_{BO}\frac{dz}{1+z^5} = \int_\infty^0 \frac{e^{\frac{2\pi i}{5}}\,dR}{1+(Re^{\frac{2\pi i}{5}})^5} = -e^{\frac{2\pi i}{5}}\int_0^\infty \frac{dR}{1+R^5}$$

Thus

$$\begin{aligned}\lim_{R\to\infty}\int_C \frac{dz}{1+z^5} &= \int_0^\infty \frac{dx}{1+x^5} - e^{\frac{2\pi i}{5}}\int_0^\infty\frac{dR}{1+R^5} = \frac{2\pi i}{5e^{\frac{4\pi i}{5}}}\\ \Rightarrow \int_0^\infty \frac{dx}{1+x^5} &= \frac{2\pi i}{5e^{\frac{4\pi i}{5}}}\frac{1}{1-e^{\frac{2\pi i}{5}}} = \frac{2\pi i}{5e^{\frac{4\pi i}{5}}}\frac{e^{-\frac{\pi i}{5}}}{e^{-\frac{\pi i}{5}}-e^{\frac{\pi i}{5}}}\\ &= \frac{\pi}{5}\frac{2i}{e^{\frac{\pi i}{5}}-e^{-\frac{\pi i}{5}}} = \frac{\pi}{5}\Big/\sin\frac{\pi}{5}\end{aligned}$$

Note: We have provided both proofs because sometimes the examiner prescribes the contour. ∎

Question 2(a) *Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be analytic for $|z| < 1+\delta, (\delta > 0)$. Prove that the polynomial $p_k(z)$ of degree k which minimizes the integral*

$$\frac{1}{2\pi}\int_0^{2\pi} |f(e^{i\theta}) - p_k(e^{i\theta})|^2\,d\theta$$

is $p_k(z) = \sum_{n=0}^k a_n z^n$. Prove that the minimum value is given by $\sum_{n=k+1}^\infty |a_n|^2$.

Solution. On $|z| = 1, z = e^{i\theta}$ and

$$\int_0^{2\pi} f(z)\overline{f(z)}\,d\theta = \int_0^{2\pi}\sum_{n,m=0}^\infty a_n\overline{a_m}e^{i(n-m)\theta}\,d\theta$$

Now termwise integration is justified because the series $\sum_{n=0}^\infty a_n z^n$ is uniformly convergent in $|z| \leq 1$ as the given series is convergent in $|z| < 1+\delta$ with $\delta > 0$. Thus

$$\frac{1}{2\pi}\int_0^{2\pi}|f(z)|^2\,d\theta = \frac{1}{2\pi}\sum_{n,m=0}^\infty a_n\overline{a_m}\int_0^{2\pi} e^{i(n-m)\theta}\,d\theta = \sum_0^\infty |a_n|^2$$

as $\int_0^{2\pi} e^{i(n-m)\theta}\,d\theta = 0$ or 2π according as $n \neq m$ or $n = m$.

Let $p_k(z) = \sum_{n=0}^k b_n z^n$, then as above

$$\frac{1}{2\pi}\int_0^{2\pi}|f(z) - p_k(z)|^2\,d\theta = \sum_{n=0}^k |a_n - b_n|^2 + \sum_{n=k+1}^\infty |a_n|^2$$

Clearly the right hand side is minimum if and only if $\sum_{n=0}^k |a_n - b_n|^2 = 0 \Rightarrow a_n = b_n$ for $n = 1, \dots, k$, as all terms in the sum are non-negative. Thus $p_k(z) = \sum_{n=0}^k a_n z^n$ and the minimum value of the integral is $\sum_{n=k+1}^\infty |a_n|^2$. ∎

Question 2(b) *If f is regular in the whole plane and the values of $f(z)$ do not lie in the disc with center w_0 and radius δ, show that f is constant.*

Solution. Liouville's Theorem: If $f(z)$ is entire, i.e. regular in the whole plane, and bounded, then $f(z)$ is constant.

Consider the function $F(z) = \frac{1}{f(z)-w_0}$. Since $f(z)$ is entire and $f(z) \neq w_0$ (note that if $f(z) = w_0$ for some z then one of its values would lie inside the disc with center w_0 and radius δ). it follows that $F(z)$ is an entire function. Since $|f(z) - w_0| > \delta$ for every z, $|F(z)| < \frac{1}{\delta}$ for every z, thus by Liouville's theorem $F(z) \equiv c$ a constant, and therefore $f(z)$ is a constant.

Proof of Liouville's theorem: From Cauchy's integral formula, we have for any z_0 and ρ however large

$$f'(z_0) = \frac{1}{2\pi i}\int_{|z-z_0|=\rho} \frac{f(z)\,dz}{(z-z_0)^2}$$

Now $f(z)$ is bounded, say $|f(z)| \leq M$ and $|z - z_0| = \rho$, so let $z - z_0 = \rho e^{i\theta}, dz = \rho i e^{i\theta}\,d\theta$ which gives us

$$|f'(z_0)| \leq \frac{M}{2\pi\rho^2}2\pi\rho = \frac{M}{\rho}$$

Letting $\rho \to \infty$, we get $f'(z_0) = 0$ for any z_0, thus $f'(z) = 0$ so f is a constant. ∎

Question 2(c) *Find the singularities of* $\sin(\frac{1}{1-z})$ *in the complex plane.*

Solution. Since $\frac{1}{1-z}$ is analytic everywhere except $z = 1$, $\sin(\frac{1}{1-z})$ is regular everywhere except $z = 1$. At $z = 1$ the function has an essential singularity — Clearly $\sin(\frac{1}{1-z}) = 0 \Leftrightarrow \frac{1}{1-z} = n\pi, n \neq 0 \Leftrightarrow z = 1 - \frac{1}{n\pi}$, $n \in \mathbb{Z}, n \neq 0$. Thus 1 is a limit point of zeros of $\sin(\frac{1}{1-z})$ and therefore $\sin(\frac{1}{1-z})$ has an essential singularity at $z = 1$.

Note that $\sin(\frac{1}{1-z})$ is regular at ∞ as $\sin(\frac{\zeta}{1-\zeta})$ is regular at $\zeta = 0$. ∎