UPSC Civil Services Main 1990 - Mathematics Complex Analysis

Brij Bhooshan

Asst. Professor

B.S.A. College of Engg & Technology

Mathura

Question 1(a) *Let f be regular for $|z| < R$, prove that, if $0 < r < R$,*

$$f'(0) = \frac{1}{\pi r}\int_0^{2\pi} u(\theta)\exp(-i\theta)\,d\theta$$

where $u(\theta) = \operatorname{Re} f(re^{i\theta})$.

Solution. Using Cauchy's integral formula, it is easily deduced that for any z in the interior of $\{C_R \ : \ |z| = R\}$, we have

$$\frac{f^{(t)}(z)}{t!} = \frac{1}{2\pi i}\int_{C_R}\frac{f(\zeta)}{(\zeta - z)^{t+1}}\,d\zeta$$

In particular, $f'(0) = \dfrac{1}{2\pi i}\displaystyle\int_{C_R}\frac{f(\zeta)}{\zeta^2}\,d\zeta$.

Putting $\zeta = Re^{i\theta}, d\zeta = Rie^{i\theta}\,d\theta$, we get

$$f'(0) = \frac{1}{2\pi i}\int_0^{2\pi}\frac{f(Re^{i\theta})}{R^2e^{2i\theta}}Rie^{i\theta}\,d\theta = \frac{1}{2\pi R}\int_0^{2\pi} f(Re^{i\theta})e^{-i\theta}\,d\theta \qquad (1)$$

We now consider the integral

$$\frac{1}{2\pi i}\int_{C_R}\frac{f(\zeta)\zeta^{t-1}}{(R - \frac{\bar{z}}{R}\zeta)^{t+1}}\,d\zeta$$

By Cauchy's residue theorem, the above integral is equal to $2\pi i$(sum of residues of the integrand within C_R). If $t \geq 1$, the only possibility of a pole could be at the point $\zeta = \frac{R^2}{\bar{z}}$, but $|z| = |\overline{z}| < R$, therefore $|\frac{R^2}{\overline{z}}| > \frac{R^2}{R} = R$, so $\frac{R^2}{\overline{z}}$ lies outside C_R and hence the integrand has no pole inside C_R, so

$$\frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)\zeta^{t-1}}{(R - \frac{\overline{z}}{R}\zeta)^{t+1}}\, d\zeta = 0 \quad \text{for } t \geq 1$$

In particular, taking $t = 1, z = 0$,

$$\frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)}{R^2}\, d\zeta = 0$$

Thus we get

$$\begin{aligned} 0 &= \frac{1}{2\pi R}\int_0^{2\pi} f(Re^{i\theta})e^{i\theta}\, d\theta \\ \Rightarrow 0 &= \frac{1}{2\pi R}\int_0^{2\pi} \overline{f}(Re^{i\theta})e^{-i\theta}\, d\theta \qquad (2) \end{aligned}$$

Adding $(1), (2)$, we get

$$f'(0) = \frac{1}{2\pi R}\int_0^{2\pi} (f(Re^{i\theta}) + \overline{f}(Re^{i\theta}))e^{-i\theta}\, d\theta = \frac{1}{\pi R}\int_0^{2\pi} u(\theta)\exp(-i\theta)\, d\theta$$

as required.

Note 1: To get the desired form, we could have considered the integral over $\{C_r : |z| = r < R\}$ instead of C_R and in that case $\zeta = re^{i\theta}$ and instead of R, we would have got r i.e.

$$f'(0) = \frac{1}{\pi r}\int_0^{2\pi} u(\theta)\exp(-i\theta)\, d\theta$$

Note 2: The integral

$$\frac{1}{2\pi i}\int_{C_R} \frac{f(\zeta)\zeta^{t-1}}{(R - \frac{\overline{z}}{R}\zeta)^{t+1}}\, d\zeta$$

plays an important role in questions of this type, and has to be kept in mind. ∎

Question 1(b) *Prove that the distance from the origin to the nearest zero of* $f(z) = \sum_{n=0}^{\infty} a_n z^n$ *is at least* $\dfrac{r|a_o|}{M + |a_0|}$ *where* r *is any number not exceeding the radius of convergence of the series, and* $M = M(r) = \sup_{|z|=r} |f(z)|$.

Solution. By Cauchy's integral formula,

$$f(z) - f(0) = \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(\zeta)\, d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{|\zeta|=r} \frac{f(\zeta)\, d\zeta}{\zeta}$$

where $|z| < r \leq R$, R is the radius of convergence. If $f(z) = 0$, then

$$|f(0)| \leq \frac{1}{2\pi} M \left| \int_{|\zeta|=r} \Big(\frac{1}{\zeta - z} - \frac{1}{\zeta}\Big)\, d\zeta \right| \leq \frac{M}{2\pi}|z| \int_0^{2\pi} \left| \frac{rie^{i\theta}\, d\theta}{re^{i\theta}(r - |z|)} \right| = \frac{M|z|}{r - |z|}$$

because $|\zeta - z| \geq |\zeta| - |z| = r - |z|$ on $|\zeta| = r$. Thus $r|f(0)| \leq |z|(M + |f(0)|) \Rightarrow |z| \geq \frac{|f(0)|r}{M + |f(0)|}$. Here $f(0) = a_0$, and the result follows. ∎

Question 1(c) *If $f = u + iv$ is regular throughout the complex plane, and $au + bv - c \geq 0$ for suitable constants a, b, c then f is constant.*

Solution. Theorem: If $f(z) = u + iv$ is entire, and $u \leq 0$, then f is constant.

Proof: Consider $F(z) = e^{f(z)}$, then $F(z)$ is also entire. Moreover

$$|F(z)| = |e^{u+iv}| = |e^u| \leq 1 \because u \leq 0$$

Thus $F(z)$ is entire and bounded, hence is a constant by Liouville's theorem. Now $F'(z) = f'(z)e^{f(z)} = 0 \Rightarrow f'(z) = 0$ because $e^{f(z)} \neq 0$, so $f(z)$ is constant.

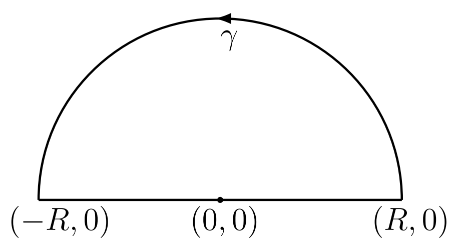
Corollary: If $f(z) = u + iv$ is entire, and $u \geq 0$, then f is constant. Proof: Consider $-f(z) = -u - iv$, then $-u \leq 0$ and $-f(z)$ is constant.

Now consider $F(z) = (a - ib)f(z) - c = (au + bv - c) + i(av - bu)$. Now $F(z)$ is entire, and $\operatorname{Re} F(z) = au + bv - c \geq 0$, so $F(z)$ is constant, hence $f(z)$ is constant. ∎

Question 2(a) *Prove that* $\displaystyle\int_{-\infty}^{\infty} \frac{x^4\,dx}{1 + x^8} = \frac{\pi}{\sqrt{2}} \sin\frac{\pi}{8}$ *using residue calculus.*

Solution.

We take $f(z) = \frac{z^4}{1+z^8}$ and the contour C consisting of γ a semicircle of radius R with center $(0, 0)$ lying in the upper half plane, and the line joining $(-R, 0)$ and $(R, 0)$. Finally we will let $R \to \infty$.



By Cauchy's residue theorem

$$\begin{aligned}\lim_{R\to\infty} \int_C \frac{z^4\,dz}{1 + z^8} &= \int_{-\infty}^{\infty} \frac{x^4\,dx}{1 + x^8} + \lim_{R\to\infty} \int_\gamma \frac{z^4\,dz}{1 + z^8} \\ &= 2\pi i(\text{sum of residues at poles of } f(z) \text{ in the upper half plane})\end{aligned}$$

Now

$$\left|\int_\gamma \frac{z^4\,dz}{1 + z^8}\right| \leq \left|\int_0^\pi \frac{R^4 e^{4i\theta} Rie^{i\theta}\,d\theta}{R^8 - 1}\right| \leq \frac{\pi R^5}{R^8 - 1}$$

because $|z^8 + 1| \geq |z^8| - 1 = R^8 - 1$ on $|z| = R$. Therefore

$$\lim_{R\to\infty} \int_\gamma \frac{z^4\,dz}{1 + z^8} = 0$$

$f(z)$ has poles at zeros of $z^8 + 1 = 0 \Rightarrow z^8 = -1 \Rightarrow z^8 = e^{(2n+1)\pi i} \Rightarrow z = e^{\frac{(2n+1)\pi i}{8}}, n \in \mathbb{Z}$. Clearly $z = e^{\frac{\pi i}{8}}, e^{\frac{3\pi i}{8}}, e^{\frac{5\pi i}{8}}, e^{\frac{7\pi i}{8}}$ are the only poles of $f(z)$ in the upper half plane and all these are simple poles. The residue at any simple pole z_0 is $\frac{z_0^4}{8z_0^7} = \frac{1}{8z_0^3}$,

$$\begin{aligned} & \text{sum of residues at poles of } f(z) \text{ in the upper half plane} \\ = & \frac{1}{8}\big(e^{-3\pi i/8} + e^{-9\pi i/8} + e^{-15\pi i/8} + e^{-21\pi i/8}\big) \\ = & \frac{1}{8}\big(e^{-3\pi i/8} - e^{-\pi i/8} + e^{\pi i/8} - e^{3\pi i/8}\big) \\ = & \frac{1}{8}\big(2i\sin\frac{\pi}{8} - 2i\sin\frac{3\pi}{8}\big) \\ = & \frac{i}{4}\big(\sin\frac{\pi}{8} - \cos\frac{\pi}{8}\big) \\ = & \frac{i\sqrt{2}}{4}(\cos\frac{\pi}{4}\sin\frac{\pi}{8} - \cos\frac{\pi}{8}\sin\frac{\pi}{4}) \\ = & -\frac{i}{2\sqrt{2}}\sin\frac{\pi}{8} \end{aligned}$$

Thus

$$\int_{-\infty}^{\infty} \frac{x^4\,dx}{1+x^8} = 2\pi i(-\frac{i}{2\sqrt{2}}\sin\frac{\pi}{8}) = \frac{\pi}{\sqrt{2}}\sin\frac{\pi}{8}$$

as required. ∎

Question 2(b) *Derive a series expansion of* $\log(1+e^z)$ *in powers of* z.

Solution. Let $f(z) = \log(1+e^z)$, then

$$f'(z) = \frac{e^z}{1+e^z} = \frac{1}{2}e^{\frac{z}{2}}\frac{2}{e^{\frac{z}{2}} + e^{-\frac{z}{2}}} = \frac{1}{2}e^{\frac{z}{2}}\frac{1}{\cosh\frac{z}{2}}$$

Let $g(z) = \cosh\frac{z}{2}$, then

$$g^{(n)}(z) = \begin{cases} \frac{1}{2^n}\sinh\frac{z}{2}, & n \text{ odd} \\ \frac{1}{2^n}\cosh\frac{z}{2}, & n \text{ even} \end{cases}$$

In particular, $g^{(n)}(0) = 0$ when n is odd, and $g^{(n)}(0) = \frac{1}{2^n}$ when n is even. Moreover

$$f'(z)\cosh\frac{z}{2} = f'(z)g(z) = \frac{1}{2}e^{\frac{z}{2}}$$

Using Leibnitz rule for the derivative of the product of two functions, we get

$$\frac{d^n}{dz^n}\left(\frac{1}{2}e^{\frac{z}{2}}\right) = \frac{e^{\frac{z}{2}}}{2^{n+1}} = \sum_{p=0}^{n}\binom{n}{p}g^{(n-p)}(z)f^{(p+1)}(z)$$

Thus when $z = 0$, we get

$$\sum_{p=0}^{n}\binom{n}{p}\frac{\epsilon_{n-p}}{2^{n-p}}f^{(p+1)}(0) = \frac{1}{2^{n+1}} \quad \text{where } \epsilon_n = \begin{cases} 0, & n \text{ odd} \\ 1, & n \text{ even} \end{cases}$$

and therefore

$$2^{n+1}f^{(n+1)}(0) = 1 - \sum_{p=0}^{n-1}\binom{n}{p}2^{p+1}\epsilon_{n-p}f^{(p+1)}(0)$$

Case (1) : When n is even

$$2^{n+1}f^{(n+1)}(0) = 1 - \binom{n}{0}2f'(0) - \sum_{p=1}^{n-2}\binom{n}{p}2^{p+1}\epsilon_{n-p}f^{(p+1)}(0)$$

Note that odd p do not contribute anything to the summation, as $\epsilon_{n-p} = 0$ for odd p. Now we can see by induction that $f^{(n)}(0) = 0$ whenever n is odd and $n > 1$. $f'(0) = \frac{1}{2}$. $2^3 f^{(3)}(0) = 1 - 2\cdot\frac{1}{2} = 0$. Assume by induction hypothesis that $f^{(3)}(0) = f^{(5)}(0) = \ldots = f^{(2m-1)}(0) = 0$, then letting $n = 2m$ in the above formula,

$$2^{2m+1}f^{(2m+1)}(0) = -\sum_{p=1}^{m-1}\binom{2m}{2p}2^{2p+1}f^{(2p+1)}(0) = 0$$

Case (2): When n is odd: The terms with even p in the formula above do not make any contribution. Thus letting $n = 2m+1$,

$$2^{2m+2}f^{(2m+2)}(0) = 1 - \sum_{r=0}^{m-1}\binom{2m+1}{2r+1}2^{2r+2}f^{(2r+2)}(0) = 1 - \sum_{r=1}^{m}\binom{2m+1}{2r-1}2^{2r}f^{(2r)}(0) \qquad (*)$$

We can now see that $f''(0) = \frac{1}{4}, f^{(4)}(0) = -\frac{1}{8}, f^{(6)}(0) = \frac{1}{4}$.

Thus

$$\begin{aligned}\log(1+e^z) &= \log 2 + \frac{z}{2} + \frac{1}{4}\frac{1}{2!}z^2 - \frac{1}{8}\frac{1}{4!}z^4 + \frac{1}{4}\frac{1}{6!}z^6 + \ldots \\ &= \log 2 + \frac{z}{2} + \sum_{n=1}^{\infty}\frac{f^{(2n)}(0)z^{2n}}{(2n)!}\end{aligned}$$

where $f^{(2n)}(0)$ is given by $(*)$ for $n \geq 1$. ∎

Note: We now present an alternative solution, where we use Leibnitz rule for the n-th derivative of the quotient of two functions. It is a good exercise in itself and is usually missing from textbooks.

Theorem: Let $y = \frac{u}{v}$, where u, v are functions with derivatives up to order n. Then

$$y_n = \frac{1}{v^{n+1}}\begin{vmatrix} v & 0 & 0 & \ldots & u \\ v_1 & v & 0 & \ldots & u_1 \\ v_2 & \binom{2}{1}v_1 & v & \ldots & u_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ v_n & \binom{n}{1}v_{n-1} & \binom{n}{2}v_{n-2} & \ldots & u_n \end{vmatrix}$$

Here the determinant is $(n+1)\times(n+1)$, and $y_n = \dfrac{d^n y}{dx^n}$.

Proof: $vy = u$, therefore, by taking successive derivatives using Leibnitz product rule we get

$$\begin{array}{lcl} vy & = & u \\ v_1 y + v y_1 & = & u_1 \\ v_2 y + 2v_1 y_1 + v y_2 & = & u_2 \\ \ldots & & \ldots \\ v_n y + \binom{n}{1} v_{n-1} y_1 + \ldots + v y_n & = & u_n \end{array}$$

These are $n+1$ equations in $n+1$ unknowns $y, y_1, \ldots, y_n$, and the determinant of the coefficient matrix is v^{n+1}. Thus by Cramer's rule

$$y_n = \frac{1}{v^{n+1}} \begin{vmatrix} v & 0 & 0 & \ldots & u \\ v_1 & v & 0 & \ldots & u_1 \\ v_2 & \binom{2}{1} v_1 & v & \ldots & u_2 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ v_n & \binom{n}{1} v_{n-1} & \binom{n}{2} v_{n-2} & \ldots & u_n \end{vmatrix}$$

as required.

Now $f(z) = \log(1+e^z), f(0) = \log 2$. $f'(z) = \frac{e^z}{1+e^z}, f'(0) = \frac{1}{2}$. Let $u = e^z, v = 1 + e^z$. Then $u_n(0) = 1$ for every n, and $v(0) = 2, v_n(0) = 1$ for $n \geq 1$. Let $F(z) = \frac{u}{v}$, then

$$F^{(n)}(0) = f^{(n+1)}(0) = \frac{1}{2^{n+1}} \begin{vmatrix} 2 & 0 & 0 & \ldots & 0 & 1 \\ 1 & 2 & 0 & \ldots & 0 & 1 \\ 1 & 2 & 2 & \ldots & 0 & 1 \\ \ldots & & & \ldots & & \ldots \\ 1 & \binom{n}{1} & \binom{n}{2} & \ldots & \binom{n}{n-1} & 1 \end{vmatrix}$$

$$\begin{array}{lclcl} F^{(1)}(0) & = & f^{(2)}(0) & = & \frac{1}{4}\begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} = \frac{1}{4} \\ F^{(2)}(0) & = & f^{(3)}(0) & = & \frac{1}{8}\begin{vmatrix} 2 & 0 & 1 \\ 1 & 2 & 1 \\ 1 & 2 & 1 \end{vmatrix} = 0 \\ F^{(3)}(0) & = & f^{(4)}(0) & = & \frac{1}{16}\begin{vmatrix} 2 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 \\ 1 & 2 & 2 & 1 \\ 1 & 3 & 3 & 1 \end{vmatrix} = \frac{1}{16}\begin{vmatrix} 2 & 0 & 0 & 1 \\ -1 & 2 & 0 & 0 \\ -1 & 2 & 2 & 0 \\ -1 & 3 & 3 & 0 \end{vmatrix} = \frac{-2}{16} = -\frac{1}{8} \\ F^{(4)}(0) & = & f^{(5)}(0) & = & \frac{1}{32}\begin{vmatrix} 2 & 0 & 0 & 0 & 1 \\ 1 & 2 & 0 & 0 & 1 \\ 1 & 2 & 2 & 0 & 1 \\ 1 & 3 & 3 & 2 & 1 \\ 1 & 4 & 6 & 4 & 1 \end{vmatrix} = 0 \end{array}$$

Thus $\log(1+e^z)$ has the expansion as given above.

Question 2(c) *Determine the nature of singular points of* $\sin\left(\dfrac{1}{\cos\frac{1}{z}}\right)$ *and investigate its behavior at* $z = \infty$.

Solution.

1. Let $\zeta = \frac{1}{z}$, and $\phi(\zeta) = f(\frac{1}{\zeta}) = \sin\left(\dfrac{1}{\cos\zeta}\right)$. Therefore $\lim\limits_{\zeta\to 0}\phi(\zeta) = \sin 1$, showing that $\phi(\zeta)$ has a removable singularity at $\zeta = 0$. In fact $\phi(\zeta)$ is analytic at $\zeta = 0$ if $\phi(0)$ is defined to be $\sin 1$. Note that

$$\lim_{\zeta\to 0}\frac{\phi(\zeta)-\phi(0)}{\zeta} = \lim_{\zeta\to 0}\frac{\sin(\frac{1}{\cos\zeta}) - \sin 1}{\zeta} = \lim_{\zeta\to 0}\cos\left(\frac{1}{\cos\zeta}\right)\sec\zeta\tan\zeta = 0$$

   Thus $\sin\left(\dfrac{1}{\cos\frac{1}{z}}\right)$ is regular at ∞.

2. At all zeros of $\cos\frac{1}{z}$ i.e. $z = \dfrac{2}{(2n+1)\pi}$ the function $\sin\left(\dfrac{1}{\cos\frac{1}{z}}\right)$ has essential singularities because $\lim_{x\to\infty}\sin x$ does not exist — if it did, then given $\epsilon > 0$, we would have N such that $x_1 > N, x_2 > N \Rightarrow |\sin x_1 - \sin x_2| < \epsilon$. But for any N we can take $x_1 = 2n\pi + \frac{\pi}{2} > x_2 = 2n\pi > N$, then $|\sin x_1 - \sin x_2| = 1 \not< \epsilon$ if $\epsilon < 1$.

3. $z = 0$ is also an essential singularity of the given function as it is a limit point of essential singularities $z = \dfrac{2}{(2n+1)\pi}$.

∎