UPSC Civil Services Main 1991 - Mathematics
Complex Analysis

Brij Bhooshan

Asst. Professor

B.S.A. College of Engg & Technology

Mathura

Question 1(a) *A function $f(z)$ is defined for all finite values of z by $f(0) = 0$ and $f(z) = e^{-z^{-4}}$ everywhere else. Show that the Cauchy-Riemann equations are satisfied at the origin. Show also that $f(z)$ is not analytic at the origin.*

Solution. Let $f(z) = u + iv$. By definition

$$\frac{\partial u}{\partial x}(0,0) = \lim_{x\to 0} \frac{u(x,0) - u(0,0)}{x}, \ \ \frac{\partial u}{\partial y}(0,0) = \lim_{y\to 0} \frac{u(0,y) - u(0,0)}{y}$$

Now $u(x,0) = \text{Re}\, f(x,0) = e^{-x^{-4}}, u(0,y) = \text{Re}\, f(0,y) = e^{-(iy)^{-4}}$, therefore

$$\frac{\partial u}{\partial x}(0,0) = \lim_{x\to 0} \frac{e^{-x^{-4}} - 0}{x} = \lim_{t\to\infty} te^{-t^4} = 0, \quad \frac{\partial u}{\partial y}(0,0) \lim_{y\to 0} \frac{e^{-y^{-4}}}{y} = 0$$

(Note that $e^{t^4} > t^4 \Rightarrow te^{-t^4} < \frac{1}{t^3} \Rightarrow \lim_{t\to\infty} te^{-t^4} = 0$).

It is obvious that $v(x,0)$ =Imaginary part of $e^{-x^{-4}} = 0$, and $v(0,y)$ =Imaginary part of $e^{-(iy)^{-4}} = 0$, and therefore $v_x(0,0) = v_y(0,0) = 0$. Thus $\frac{\partial u}{\partial x}(0,0) = \frac{\partial v}{\partial y}(0,0), \frac{\partial u}{\partial y}(0,0) = -\frac{\partial v}{\partial x}(0,0)$, i.e. the Cauchy-Riemann equations are satisfied at $(0,0)$.

However $f(z)$ is not analytic at $z = 0$ because it is not even continuous at $z = 0$: if we take $z = re^{\frac{i\pi}{4}}$, then $z \to 0 \Leftrightarrow r \to 0$, but $\lim_{r\to 0} f(re^{\frac{i\pi}{4}}) = \lim_{r\to 0} e^{-r^{-4}e^{\pi i}} = \lim_{r\to 0} e^{r^{-4}} = \infty$, so $\lim_{z\to 0} f(z) \neq f(0)$. ∎

Question 1(b) *If $|a| \neq R$, show that*

$$\int_{|z|=R} \frac{|dz|}{|(z-a)(z+a)|} < \frac{2\pi R}{|R^2 - |a|^2|}$$

Solution. On $|z| = R, z = Re^{i\theta}, 0 \leq \theta \leq 2\pi, |dz| = |Rie^{i\theta}\,d\theta| = R\,d\theta$. $|z^2 - a^2| \geq |z|^2 - |a|^2 = R^2 - |a|^2$ and $|z^2 - a^2| \geq |a|^2 - |z|^2 = |a|^2 - R^2$, showing that $|z^2 - a^2| \geq |R^2 - |a|^2|$, with the strict inequality occurring when $a = |a|e^{i\theta}, z \neq Re^{i\theta}$. Thus

$$\int_{|z|=R} \frac{|dz|}{|(z-a)(z+a)|} < \int_0^{2\pi} \frac{R\,d\theta}{|R^2 - |a|^2|} = \frac{2\pi R}{|R^2 - |a|^2|}$$

as required. ∎

Question 1(c) *If*

$$J_n(t) = \frac{1}{2\pi} \int_0^{2\pi} \cos(n\theta - t\sin\theta)\,d\theta$$

show that

$$e^{\frac{t}{2}(z - \frac{1}{z})} = J_0(t) + zJ_1(t) + z^2 J_2(t) + \ldots - \frac{1}{z}J_1(t) + \frac{1}{z^2}J_2(t) - \frac{1}{z^3}J_3(t) + \ldots$$

Solution. The function $f(z) = e^{\frac{t}{2}(z - \frac{1}{z})}$ is analytic in $0 < |z| < \infty$ and therefore by Laurent's theorem — If $f(z)$ is analytic is the annular region $D : R_1 < |z - z_0| < R_2$ and if C is any positively oriented simple closed contour lying within the region D, then for any $z \in D$, we have

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} b_n (z - z_0)^{-n}$$

where

$$a_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}, b_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{-n+1}}$$

or

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \text{ where } c_n = \frac{1}{2\pi i} \int_C \frac{f(z)\,dz}{(z - z_0)^{n+1}}$$

If we take $f(z) = e^{\frac{t}{2}(z - \frac{1}{z})}, R_1 = 0, R_2 = \infty, z_0 = 0$, then

$$f(z) = e^{\frac{t}{2}(z - \frac{1}{z})} = \sum_{n=-\infty}^{\infty} c_n z^n \text{ where } c_n = \frac{1}{2\pi i} \int_C \frac{e^{\frac{t}{2}(z - \frac{1}{z})}\,dz}{z^{n+1}}$$

We now take C as $|z| = 1$. Then $z = e^{i\theta}$ and

$$\begin{aligned} c_n &= \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{\frac{t}{2}(e^{i\theta} - e^{-i\theta})}}{e^{i(n+1)\theta}} ie^{i\theta} d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} e^{ti\sin\theta} e^{-in\theta}\,d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \cos(n\theta - t\sin\theta)\,d\theta + \frac{i}{2\pi} \int_0^{2\pi} \sin(-n\theta + t\sin\theta)\,d\theta \end{aligned}$$

But $\int_0^{2\pi} \sin(-n\theta + t\sin\theta)\,d\theta = 0$, because if we put $\theta = 2\pi - \eta$, then $\int_0^{2\pi} \sin(-n\theta + t\sin\theta)\,d\theta = \int_{2\pi}^{0} \sin(-n\eta + t\sin\eta)\,d\eta$. Therefore

$$c_n = \frac{1}{2\pi}\int_0^{2\pi} \cos(n\theta - t\sin\theta)\,d\theta = J_n(t)$$

Hence

$$e^{\frac{t}{2}(z-\frac{1}{z})} = \sum_{n=-\infty}^{\infty} J_n(t) z^n$$

Since on replacing z by $-\frac{1}{z}$, the function $f(z)$ remains unaltered, we get $J_{-n}(t) = J_n(t)$ if n is even, and $J_{-n}(t) = -J_n(t)$ if n is odd. Thus

$$e^{\frac{t}{2}(z-\frac{1}{z})} = J_0(t) + zJ_1(t) + z^2J_2(t) + \ldots - \frac{1}{z}J_1(t) + \frac{1}{z^2}J_2(t) - \frac{1}{z^3}J_3(t) + \ldots$$

as required. ∎

Question 2(a) *Examine the nature of the singularity of e^z at ∞.*

Solution. e^z has an essential singularity at ∞. We examine the nature of the singularity of $e^{\frac{1}{\zeta}}$ at $\zeta = 0$. Taking $\zeta = \frac{1}{n}$, $\lim_{\zeta \to 0} e^{\frac{1}{\zeta}} = \lim_{n\to\infty} e^n = \infty$.

Taking $\zeta = -\frac{1}{n}$, $\lim_{\zeta \to 0} e^{\frac{1}{\zeta}} = \lim_{n\to\infty} e^{-n} = 0$.

Thus $\lim_{\zeta \to 0} e^{\frac{1}{\zeta}}$ does not exist and therefore $e^{\frac{1}{\zeta}}$ has an essential singularity at $\zeta = 0$, proving that e^z has an essential singularity at ∞.

Alternately $e^{\frac{1}{\zeta}} = \sum_{n=0}^{\infty} \frac{1}{n!}\frac{1}{\zeta^n}$ is the Laurent expansion of $e^{\frac{1}{\zeta}}$ having infinitely many negative powers, showing the same result. ∎

Question 2(b) *Evaluate the residues of the function $\frac{z^3}{(z-2)(z-3)(z-5)}$ at all its singularities and show that their sum is 0.*

Solution. The given function has simple poles at $z = 2, 3, 5$.

Residue at $z = 2$ is $\lim_{z\to 2} \frac{z^3(z-2)}{(z-2)(z-3)(z-5)} = \frac{8}{3}$.

Residue at $z = 3$ is $\lim_{z\to 3} \frac{z^3(z-3)}{(z-2)(z-3)(z-5)} = -\frac{27}{2}$.

Residue at $z = 5$ is $\lim_{z\to 5} \frac{z^3(z-5)}{(z-2)(z-3)(z-5)} = \frac{125}{6}$.

Residue at ∞ is $= -$ coefficient of $\frac{1}{z}$ in the expansion of $f(z)$ around ∞.

$$\begin{aligned} f(z) &= \left(1-\frac{2}{z}\right)^{-1}\left(1-\frac{3}{z}\right)^{-1}\left(1-\frac{5}{z}\right)^{-1} \\ &= \left(1+\frac{2}{z}+\text{Higher powers of }\frac{1}{z}\right)\left(1+\frac{3}{z}+\text{Higher powers of }\frac{1}{z}\right) \\ &\quad \left(1+\frac{5}{z}+\text{Higher powers of }\frac{1}{z}\right) \\ &= 1+\frac{10}{z}+\text{Higher powers of }\frac{1}{z} \end{aligned}$$

Thus the residue at ∞ is -10.

Sum of all residues is $\dfrac{16-81+125-60}{6} = 0$.

Note: The function $f(z)$ has no singularity as such at ∞, but the residue at ∞ is always defined as such. The function is actually analytic at ∞ as $f(\frac{1}{z})$ is analytic at $z=0$. ∎

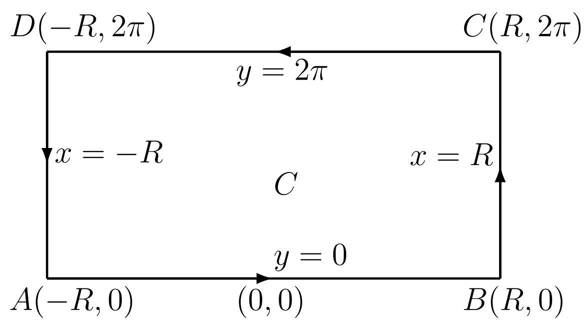
Question 2(c) *By integrating along a suitable contour show that*

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{\pi}{\sin a\pi}$$

where $0 < a < 1$.

Solution.

Our $f(z) = \frac{e^{az}}{1+e^z}$ and the contour is C, the rectangle $ABCD$ where $A = (-R, 0), B = (R, 0), C = (R, 2\pi), D = (-R, 2\pi)$ oriented in the anticlockwise direction. We let $R \to \infty$ eventually.



The function $f(z)$ has only a simple pole at $z = \pi i$ in the strip bounded by $y = 0$ and $y = 2\pi$. Residue of $f(z)$ at πi is $\displaystyle\lim_{z\to\pi i} \frac{z-\pi i}{1+e^z} e^{az} = \frac{e^{a\pi i}}{e^{\pi i}} = -e^{a\pi i}$.

Thus by Cauchy's residue theorem $\displaystyle\lim_{R\to\infty} \int_C \frac{e^{az}\,dz}{1+e^z} = -2\pi i e^{\pi i a}$.

We now evaluate the integral along the four lines.

1. On the line BC i.e. $x = R, z = R + iy, dz = i\,dy$ and

$$\left|\int_{BC} \frac{e^{az}\,dz}{1+e^z}\right| = \left|\int_0^{2\pi} \frac{e^{a(R+iy)} i\,dy}{1+e^{R+iy}}\right| \le \int_0^{2\pi} \frac{e^{aR}\,dy}{e^R-1} = \frac{2\pi e^{aR}}{e^R-1}$$

because $|e^z+1| \ge |e^z| - 1 = |e^{R+iy}| - 1 = e^R - 1$ on BC. Since $\lim_{R\to\infty} \frac{e^{aR}}{e^R-1} = 0$ as $0 < a < 1$ using L'Hospital's Rule, it follows that $\displaystyle\lim_{R\to\infty} \int_{BC} \frac{e^{az}\,dz}{1+e^z} = 0$.

2. On the line DA i.e. $x = -R, z = -R + iy, dz = i\,dy$. Since $|e^z + 1| \geq 1 - |e^z| = 1 - |e^{-R+iy}| = 1 - e^{-R}$

$$\left|\int_{DA} \frac{e^{az}\,dz}{1+e^z}\right| \leq \frac{2\pi e^{-aR}}{1-e^{-R}} (\to 0 \; as \; R \to \infty)$$

thus $\lim\limits_{R\to\infty} \int_{DA} \frac{e^{az}\,dz}{1+e^z} = 0.$

3. On the line AB, $z = x$, so

$$\lim_{R\to\infty} \int_{AB} \frac{e^{az}\,dz}{1+e^z} = \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx$$

4. On the line CD, $z = x + 2\pi i$, so

$$\lim_{R\to\infty} \int_{CD} \frac{e^{az}\,dz}{1+e^z} = \int_{\infty}^{-\infty} \frac{e^{a(x+2\pi i)}}{1+e^{x+2\pi i}}\,dx = -e^{2\pi i a} \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx$$

Thus

$$\lim_{R\to\infty} \int_{C} \frac{e^{az}\,dz}{1+e^z} = (1 - e^{2\pi i a}) \int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = -2\pi i e^{\pi i a}$$

so

$$\int_{-\infty}^{\infty} \frac{e^{ax}}{1+e^x}\,dx = \frac{-2\pi i e^{\pi i a}}{1-e^{2\pi i a}} = \frac{-2\pi i}{e^{-\pi i a} - e^{\pi i a}} = \frac{\pi}{\sin a\pi}$$

as required. ∎