UPSC Civil Services Main 1992 - Mathematics Complex Analysis

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Question 1(a) *If $u = e^{-x}(x \sin y - y \cos y)$, find v such that $f(z) = u + iv$ is analytic. Also find $f(z)$ explicitly as a function of z.*

Solution. See 1993, question 2(b). ∎

Question 1(b) *Let $f(z)$ be analytic inside and on the circle C defined by $|z| = R$ and let $re^{i\theta}$ be any point inside C. Prove that*

$$f(re^{i\theta}) = \frac{1}{2\pi} \int_0^{2\pi} \frac{(R^2 - r^2) f(Re^{i\phi})}{R^2 - 2Rr\cos(\theta - \phi) + r^2} \, d\phi$$

Solution. By Cauchy's integral formula

$$f(z) = f(re^{i\theta}) = \frac{1}{2\pi i} \int_{C_R : |\zeta| = r} \frac{f(\zeta)}{\zeta - z} \, d\zeta \qquad (1)$$

We note that the function $\dfrac{f(\zeta)}{\zeta - \frac{R^2}{\overline{z}}}$ has no singularity within and on C_R, because $f(\zeta)$ is analytic within and on C_R and $(\zeta - \frac{R^2}{\overline{z}})^{-1}$ is also analytic within and on C_R as $\frac{R^2}{\overline{z}}$ lies outside C_R and therefore $\zeta - \frac{R^2}{\overline{z}} \neq 0$ (Note that $R^2 = R \cdot R > R|\overline{z}|$, because $|z| = r < R$, thus $|\frac{R^2}{\overline{z}}| > R$. Thus by Cauchy's theorem

$$0 = \int_{C_R} \frac{f(\zeta)}{\zeta - \frac{R^2}{\overline{z}}} \, d\zeta \qquad (2)$$

Using $(1),(2)$ we get

$$\begin{aligned}
f(z) &= \frac{1}{2\pi i}\int_{|\zeta|=r} f(\zeta)\left[\frac{1}{\zeta - z} - \frac{1}{\zeta - \frac{R^2}{\overline{z}}}\right] d\zeta \\
&= \frac{1}{2\pi i}\int_{|\zeta|=r} f(\zeta)\left[\frac{z - \frac{R^2}{\overline{z}}}{(\zeta - z)(\zeta - \frac{R^2}{\overline{z}})}\right] d\zeta \\
&= \frac{1}{2\pi i}\int_{|\zeta|=r} f(\zeta)\left[\frac{z\overline{z} - R^2}{(\zeta - z)(\zeta\overline{z} - R^2)}\right] d\zeta \\
\Rightarrow f(re^{i\theta}) &= \frac{1}{2\pi i}\int_0^{2\pi} f(Re^{i\phi})\left[\frac{r^2 - R^2}{(Re^{i\phi} - re^{i\theta})(rRe^{i(\phi-\theta)} - R^2)}\right] Re^{i\phi} i\, d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\phi})\left[\frac{r^2 - R^2}{(R - re^{i(\theta-\phi)})(re^{i(\phi-\theta)} - R)}\right] d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\phi})\left[\frac{r^2 - R^2}{-R^2 - r^2 + rR(e^{i(\theta-\phi)} + e^{i(\phi-\theta)})}\right] d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\phi})\left[\frac{R^2 - r^2}{R^2 + r^2 + 2rR\cos(\theta - \phi)}\right] d\phi
\end{aligned}$$

as required. ∎

Question 1(c) *Prove that all the roots of* $z^7 - 5z^3 + 12 = 0$ *lie between the circles* $|z| = 1$ *and* $|z| = 2$.

Solution. See 2006 question 2(b). ∎

Question 2(a) *Find the region of convergence of the series whose* n*-th term is* $\dfrac{(-1)^{n-1}z^{2n-1}}{(2n-1)!}$.

Solution. Clearly

$$\left|\frac{\text{Coefficient of the } (n+1)\text{-th term}}{\text{Coefficient of the } n\text{-th term}}\right| = \frac{(2n-1)!}{(2n+1)!} \to 0 \text{ as } n \to \infty$$

Thus $\lim_{n\to\infty} |\text{Coefficient of the } n\text{-th term}|^{\frac{1}{n}} = 0$. So the radius of convergence of the power series $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}z^{2n-1}}{(2n-1)!}$ is ∞, i.e. the region of convergence is the entire complex plane. ∎

Question 2(b) *Expand* $f(z) = \dfrac{1}{(z+1)(z+3)}$ *in a Laurent series valid for* (i) $|z| > 3$, (ii) $1 < |z| < 3$, (iii) $|z| < 1$.

Solution. (i) $|z| > 3$.

$$f(z) = \frac{1}{2}\Big(\frac{1}{z+1} - \frac{1}{z+3}\Big) = \frac{1}{2z}\Big[\Big(1+\frac{1}{z}\Big)^{-1} - \Big(1+\frac{3}{z}\Big)^{-1}\Big]$$

Since $|\frac{1}{z}| < \frac{1}{3}, |\frac{3}{z}| < 1$, we have

$$\begin{aligned} f(z) &= \frac{1}{2z}\Big[\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} - \sum_{n=0}^{\infty}\frac{(-1)^n 3^n}{z^n}\Big] \\ &= \frac{1}{2z}\sum_{n=0}^{\infty}\frac{(-1)^n(1-3^n)}{z^n} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(1-3^n)}{2}\frac{1}{z^{n+1}} \end{aligned}$$

(ii) $1 < |z| < 3$.

$$f(z) = \frac{1}{2z}\Big(1+\frac{1}{z}\Big)^{-1} - \frac{1}{2}\frac{1}{3}\Big(1+\frac{z}{3}\Big)^{-1}$$

Since $|\frac{1}{z}| < 1, |\frac{z}{3}| < 1$, we get

$$\begin{aligned} f(z) &= \frac{1}{2z}\sum_{n=0}^{\infty}\frac{(-1)^n}{z^n} - \frac{1}{6}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^n} \\ &= \frac{1}{2}\Big[\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{n+1}} + \sum_{n=0}^{\infty}\frac{(-1)^{n+1}z^n}{3^{n+1}}\Big] \end{aligned}$$

(iii) $|z| < 1$.

$$f(z) = \frac{1}{2}\Big(1+z\Big)^{-1} - \frac{1}{2}\frac{1}{3}\Big(1+\frac{z}{3}\Big)^{-1}$$

As $|z| < 1, |\frac{z}{3}| < 1$, we get

$$\begin{aligned} f(z) &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{6}\sum_{n=0}^{\infty}\frac{(-1)^n z^n}{3^n} \\ &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\Big(1-\frac{1}{3^{n+1}}\Big)z^n \end{aligned}$$

These are the Laurent or Taylor series in the required three cases. ∎

Question 2(c) *By integrating along a suitable contour evaluate* $\displaystyle\int_0^{\infty}\frac{\cos mx}{x^2+1}\,dx$

Solution. See 1995, question 2(a). ∎