UPSC Civil Services Main 1993 - Mathematics Complex Analysis

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Question 1(a) *In the finite plane, show that the function* $f(z) = \sec\dfrac{1}{z}$ *has infinitely many isolated singularities in a finite interval which includes zero.*

Solution. We know that $\cos\dfrac{1}{z} = 0$ if and only if $\dfrac{1}{z} = (2n+1)\dfrac{\pi}{2}, n \in \mathbb{Z}$, or $z = \dfrac{2}{(2n+1)\pi}$. Moreover all these zeros are simple zeros of $\cos\dfrac{1}{z}$ and are isolated singular points. Thus the given function has infinitely many simple poles at the points $z = \dfrac{2}{(2n+1)\pi}$. Since $\dfrac{2}{(2n+1)\pi} \to 0$ as $n \to \infty$, it follows that any finite interval containing 0 will have all but finitely many points of the type $z = \dfrac{2}{(2n+1)\pi}$. Thus any finite interval containing 0 will have infinitely many isolated singularities (simple poles) of $\sec\dfrac{1}{z}$. ∎

Question 1(b) *Find the orthogonal trajectories of the family of curves in the xy-plane defined by* $e^{-x}(x\sin y - y\cos y) = \alpha$, *where* α *is a real constant.*

Solution. If $f(z) = u + iv$ is an analytic function, then $u =$constant, $v =$constant represent families of curves which are orthogonal to each other, because of the Cauchy-Riemann equations:

$$\left(-\frac{\partial u}{\partial x}\Big/\frac{\partial u}{\partial y}\right) \times \left(-\frac{\partial v}{\partial x}\Big/\frac{\partial v}{\partial y}\right) = -1$$

i.e. tangents to the curve $u = c$ and $v = c'$ respectively cut each other at right angles. Thus given $u = e^x(x\sin y - y\cos y)$ we have to find v so that $f = u + iv$ is analytic.

We use Milne Thompson's method. We know

$$f'(z) = \frac{\partial u}{\partial x}\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right) - i\frac{\partial u}{\partial y}\left(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\right)$$

is an identity.

$$f'(z) = e^{-x}\sin y - e^{-x}(x\sin y - y\cos y) - ie^{-x}(x\cos y - \cos y + y\sin y)$$

Putting $z = \overline{z} \Rightarrow x = z, y = 0 \Rightarrow f'(z) = -ie^{-z}(z-1)$, or $f(z) = -i\int e^{-z}(z-1)\,dz = ize^{-z}$. Thus

$$\begin{aligned} u + iv &= i(x+iy)e^{-x}(\cos y - i\sin y) \\ &= e^{-x}(x\sin y - y\cos y) + ie^{-x}(x\cos y + y\sin y) \end{aligned}$$

Thus $v = e^{-x}(x\cos y + y\sin y) = \beta$ is the required family of curves. ∎

Question 1(c) *Prove by applying Cauchy's integral formula or otherwise that*

$$\int_0^{2\pi} \cos^{2n}\theta\,d\theta = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}2\pi$$

where $n = 1, 2, 3, \ldots$.

Solution. We put $z = e^{i\theta}$ so that $dz = ie^{i\theta}\,d\theta$ and the integral is along the curve $|z| = 1$. We get

$$\int_0^{2\pi} \cos^{2n}\theta\,d\theta = \int_{|z|=1} \frac{\left(z+\frac{1}{z}\right)^{2n}}{2^{2n}}\frac{dz}{iz} = \frac{1}{i}\int_{|z|=1} \frac{(1+z^2)^{2n}}{2^{2n}z^{2n+1}}\,dz$$

Thus by Cauchy's residue theorem $\int_0^{2\pi}\cos^{2n}\theta\,d\theta = 2\pi i\frac{1}{2^{2n}i}$(sum of residues at poles of $\frac{(1+z^2)^{2n}}{z^{2n+1}}$ inside $|z| = 1$). Clearly $z = 0$ is the only pole of the integrand in $|z| = 1$, and it is of order $2n+1$.

Residue of $\frac{(1+z^2)^{2n}}{z^{2n+1}}$ at $z = 0$ is the coefficient of $\frac{1}{z}$ in the Laurent expansion.

Now $(1+z^2)^{2n}$ =sum of powers of z with exponent $< 2n$ + $\binom{2n}{n}z^{2n}$ + sum of powers with exponent $> 2n$. Thus coefficient of $\frac{1}{z}$ in the Laurent expansion of $\frac{(1+z^2)^{2n}}{z^{2n+1}}$ around $z = 0$ is $\binom{2n}{n}$. Thus

$$\int_0^{2\pi} \cos^{2n}\theta\,d\theta = 2\pi\frac{1}{2^{2n}}\frac{2n!}{n!n!}$$

Now $2^n n! = 2n(2n-2)(2n-4)\ldots\cdot 6\cdot 4\cdot 2$, so $\frac{2n!}{2^n n!} = (2n-1)(2n-3)(2n-5)\ldots\cdot 5\cdot 3\cdot 1$. Hence

$$\int_0^{2\pi} \cos^{2n}\theta\,d\theta = \frac{1\cdot 3\cdot 5\cdot \ldots \cdot (2n-1)}{2\cdot 4\cdot 6\cdot \ldots \cdot (2n)}2\pi$$

as required. ∎

Question 2(a) *If C is the curve $y = x^3 - 3x^2 + 4x - 1$ joining the points $(1,1)$ and $(2,3)$, find the value of* $\displaystyle\int_C (12z^2 - 4iz)\,dz$.

Solution. If C_1 is any curve joining $(1,1)$ and $(2,3)$, then C and C_1 form a closed contour. Since $12z^2 - 4iz$ is analytic, by Cauchy's theorem

$$-\int_C (12z^2 - 4iz)\,dz + \int_{C_1} (12z^2 - 4iz)\,dz = 0$$

so the integral is independent of the path between $(1,1)$ and $(2,3)$. Thus

$$\begin{aligned}\int_C (12z^2 - 4iz)\,dz &= \Big[4z^3 - 2iz^2\Big]_{1+i}^{2+3i} \\ &= 4[(2+3i)^3 - (1+i)^3] - 2i[(2+3i)^2 - (1+i)^2] \\ &= 4[8 + 36i - 54 - 27i - 1 - 3i + 3 + i] - 2i[4 + 12i - 9 - 1 - 2i + 1] \\ &= 4[-44 + 7i] - 2i[-5 + 10i] = -156 + 38i\end{aligned}$$

Note that calculating $\int_C$ by $\int_C u dx + v dy$ would be more work. ∎

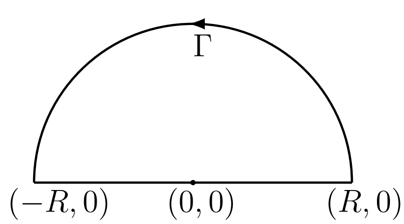
Question 2(b) *Prove that the series* $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$ *converges absolutely for* $|z| \leq 1$.

Solution. Consider the series $\displaystyle\sum_{n=1}^{\infty} a_n$ where $a_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{n+1}$. Now $s_n = \sum_{r=1}^{n}(\frac{1}{r} - \frac{1}{r+1}) = 1 - \frac{1}{n+1}$. Thus $s_n \to 1$ as $n \to \infty$, so $\displaystyle\sum_{n=1}^{\infty} a_n$ is convergent. Now $\left|\dfrac{z^n}{n(n+1)}\right| \leq a_n$ for $|z| \leq 1$, therefore by Weierstrass M-test, the series $\displaystyle\sum_{n=1}^{\infty} \frac{z^n}{n(n+1)}$ converges absolutely (in fact uniformly) in the region $|z| \leq 1$. ∎

Question 2(c) *Evaluate* $\displaystyle\int_0^{\infty} \frac{dx}{x^6 + 1}$ *by choosing an appropriate contour.*

Solution.

We take $f(z) = \frac{1}{1+z^6}$ and the contour γ consisting of Γ a semicircle of radius R with center $(0,0)$ lying in the upper half plane, and the line joining $(-R,0)$ and $(R,0)$.



By Cauchy's residue theorem $\displaystyle\int_\gamma \frac{dz}{1+z^6} = 2\pi i$(sum of residues at poles of $f(z)$ in the upper half plane).

Clearly $\frac{1}{1+z^6}$ has simple poles at $z = e^{\frac{\pi i}{6}}, z = e^{\frac{\pi i}{2}}$ and $z = e^{\frac{5\pi i}{6}}$ inside the contour.

Residue at $z = \zeta$ is $\dfrac{1}{6\zeta^5}$.

$$\begin{aligned} \text{Sum of residues} &= \frac{1}{6}\left[\frac{1}{e^{\frac{5\pi i}{6}}} + \frac{1}{e^{\frac{15\pi i}{6}}} + \frac{1}{e^{\frac{25\pi i}{6}}}\right] \\ &= \frac{1}{6}[\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6} - i + \cos\frac{\pi}{6} - i\sin\frac{\pi}{6}] = \frac{-2i}{6} \end{aligned}$$

as $\cos\frac{5\pi}{6} = -\cos\frac{\pi}{6}, \sin\frac{5\pi}{6} = \sin\frac{\pi}{6} = \frac{1}{2}$. Thus $\displaystyle\lim_{R\to\infty}\int_\gamma \frac{dz}{1+z^6} = 2\pi i\frac{-i}{3} = \frac{2\pi}{3}$.

Now

$$\left|\int_\Gamma \frac{dz}{1+z^6}\right| \leq \int_0^\pi \frac{R}{R^6-1}\,d\theta = \frac{\pi R}{R^6-1}$$

on putting $z = Re^{i\theta}$ and using $|z^6+1| \geq R^6 - 1$ on Γ.

Thus $\displaystyle\int_\Gamma \frac{dz}{1+z^6} \to 0$ as $R \to \infty$. Consequently,

$$\lim_{R\to\infty}\int_\gamma \frac{dz}{1+z^6} = \int_{-\infty}^\infty \frac{dx}{1+x^6} = \frac{2\pi}{3}$$

and hence $\displaystyle\int_0^\infty \frac{dx}{1+x^6} = \frac{\pi}{3}$. ∎