UPSC Civil Services Main 1994 - Mathematics Complex Analysis

Brij Bhooshan

Asst. Professor

B.S.A. College of Engg & Technology

Mathura

Question 1(a) *Suppose that z is the position vector of a particle moving on the ellipse $C : z = a\cos\omega t + ib\sin\omega t$ where ω, a, b are positive constants, $a > b$ and t is time. Determine where*

1. *the velocity has the greatest magnitude.*
2. *the acceleration has the least magnitude.*

Solution. See 1996, question 1(a). ∎

Question 1(b) *How many zeroes does the polynomial $p(z) = z^4 + 2z^3 + 3z + 4$ possess (i) in the first quadrant, (ii) in the fourth quadrant.*

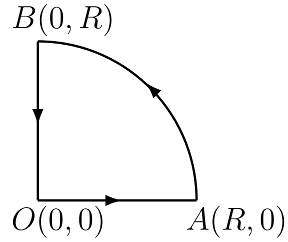
Solution.

1. $p(-1) = 0$. $p(-2) = -2 < 0, p(-3) = 22 > 0$, therefore the intermediate value theorem shows that there exists $x, -3 < x < -2$ such that $p(x) = 0$. Thus we have determined that to zeros of $p(z)$ lie on the negative real axis, and since p is a polynomial of degree 4 and hence has 4 zeros, we are left with the task of locating the the remaining two zeros.

2. $p(z)$ has no zeros on the positive real axis because $p(x) > 0$ when $x \geq 0$.

3. $p(z)$ has has no zero on the imaginary axis because $p(iy) = y^4 + 4 - 2iy^3 + 3iy = 0 \Rightarrow y^4 + 4 = 0, 2y^3 - 3y = 0$, but $y^4 + 4 = 0$ has no real zeros, so $p(iy) \neq 0$.

We now consider the contour $OABO$ where OA is straight line joining $(0,0)$ and $(R,0)$, AB is the arc of the circle $x^2+y^2=R^2$ in the first quadrant, and BO is the line joining $(0,R)$ to $(0,0)$.



By the Argument Principle, the number of zeros of $p(z)$ in the first quadrant $= \frac{1}{2\pi}\times$ (the change in the argument of $p(z)$ when z moves along the contour $OABO$ oriented anti-clockwise as $R \to \infty$).

Change in the argument along OA: On OA, $p(z) = x^4 + 2x^3 + 3x + 4 > 0 \Rightarrow \arg p(z) = 0$ for every x on OA. Therefore as z moves from O to A, the change in the argument of $p(z)$ i.e. $\Delta_{OA} \arg p(z) = 0$.

Change in the argument along BO: On BO, $z = iy$ and $p(z) = y^4 + 4 + i(3y - 2y^3)$. Therefore $\arg p(z) = \tan^{-1}\Big(\dfrac{3y-2y^3}{y^4+4}\Big)$.

$$\Delta_{BO} \arg p(z) = \tan^{-1}\Big(\frac{3y-2y^3}{y^4+4}\Big)\Big]_{\infty}^{0} = 0 - 0 = 0$$

Change in argument along AB: On arc AB, $z = Re^{i\theta}, 0 \le \theta \le \frac{\pi}{2}$, so that

$$p(z) = R^4e^{4i\theta} + 2R^3e^{3i\theta} + 3Re^{i\theta} + 4 = R^4e^{4i\theta}\Big[1 + \frac{2}{Re^{i\theta}} + \frac{3}{R^3e^{3i\theta}} + \frac{4}{R^4e^{4i\theta}}\Big] \longrightarrow R^4e^{4i\theta}$$

as $R \to \infty$. Thus $\Delta_{AB} \arg p(z) = 4\theta\Big]_0^{\frac{\pi}{2}} = 2\pi$.[1]

Hence $\Delta_{OABO} \arg p(z) = 2\pi$ as $R \to \infty$, so $p(z)$ has exactly one zero in the first quadrant.

Since $p(z)$ is a polynomial with real coefficients, it follows that if ζ is a zero of $p(z)$ and it lies in the first quadrant, then $\overline{\zeta}$ is also a zero of $p(z)$ and it lies in the fourth quadrant.

Thus $p(z)$ has one zero in each of the first and the fourth quadrants. ∎

Question 1(c) *Test for uniform convergence in the region* $|z| \le 1$ *the series*

$$\sum_{n=1}^{\infty} \frac{\cos nz}{n^3}$$

Solution. By definition

$$\cos nz = \frac{e^{inz} + e^{-inz}}{2} = \frac{e^{-ny}e^{inx} + e^{ny}e^{-inx}}{2}$$

[1]Alternately, $p(z) = z^4\Big(1 + \frac{2}{z} + \frac{3}{z^3} + \frac{4}{z^4}\Big) = z^4(1+w)$ where $w = \frac{2}{z} + \frac{3}{z^3} + \frac{4}{z^4}$. Clearly $w \to 0$ as $R \to \infty$. Therefore $|1+w-1| < \epsilon$ for $|z|$ large. This means $1+w$ remains inside a circle of radius 1 as z moves along AB and $R \to \infty$. Therefore $\Delta_{AB} \arg(1+w) = 0$ and $\Delta_{AB} p(z) = \Delta_{AB} z^4 + \Delta_{AB}(1+w) = 4\Delta_{AB} z = 4 \cdot \frac{\pi}{2} = 2\pi$.

and therefore

$$\sum_{n=1}^{\infty} \frac{\cos nz}{n^3} = \sum_{n=1}^{\infty} \frac{e^{-ny}e^{inx}}{2n^3} + \sum_{n=1}^{\infty} \frac{e^{ny}e^{-inx}}{2n^3}$$

Case 1: $y > 0$.

$$\sum_{n=1}^{\infty} \left| \frac{e^{-ny}e^{inx}}{2n^3} \right| \leq \sum_{n=1}^{\infty} \frac{1}{2n^3}$$

showing that the first term is absolutely convergent.

But the second term is not convergent, because its n-th term $\left| \dfrac{e^{ny}e^{-inx}}{2n^3} \right| \not\to 0$ as $n \to \infty$ — in fact $\left| \dfrac{e^{ny}e^{-inx}}{2n^3} \right| \to \infty$ as $n \to \infty$ when $y > 0$.

Therefore $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nz}{n^3}$ is not even convergent when $y > 0$.

Case 2: $y < 0$. This case is entirely analogous to the above case — the first term $\displaystyle\sum_{n=1}^{\infty} \frac{e^{-ny}e^{inx}}{2n^3}$ is not convergent, so $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nz}{n^3}$ is not convergent.

Case 3: $y = 0$. $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n^3}$ is uniformly and absolutely convergent, because of Weierstrass M-test, which states that if $\sum_{n=1}^{\infty} f_n(z)$ is a series and there exist positive constants M_n such that $|f_n(z)| < M_n$ for every $z \in \Omega$ and $\sum_n M_n$ is convergent, then $\sum_{n=1}^{\infty} f_n(z)$ is absolutely and uniformly convergent in Ω. Here $M_n = \frac{1}{n^3}$ for all x.

Thus the given series converges uniformly only on the real axis in $|z| \leq 1$. ∎

Question 2(a) *Find the Laurent series for*

1. $\dfrac{e^{2z}}{(z-1)^3}$ *about* $z = 1$.
2. $\dfrac{1}{z^2(z-3)^2}$ *about* $z = 3$.

Solution.

1. The function e^{2z} is analytic everywhere in the complex plane. The Taylor series of e^{2z} with center $z = 1$ is given by

$$e^{2z} = \sum_{n=0}^{\infty} \frac{\frac{d^n e^{2z}}{dz^n} \text{ at } z=1}{n!}(z-1)^n = \sum_{n=0}^{\infty} \frac{2^n e^2}{n!}(z-1)^n$$

because $\dfrac{d^n e^{2z}}{dz^n} = 2^n e^{2z}$. Thus

$$\begin{aligned}\frac{e^{2z}}{(z-1)^3} &= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{4e^2}{2!(z-1)} + \sum_{n=3}^{\infty} \frac{2^n e^2}{n!}(z-1)^{n-3} \\ &= \frac{e^2}{(z-1)^3} + \frac{2e^2}{(z-1)^2} + \frac{4e^2}{2!(z-1)} + \sum_{n=0}^{\infty} \frac{2^{n+3} e^2}{(n+3)!}(z-1)^{n}\end{aligned}$$

which is the required Laurent series of $\dfrac{e^{2z}}{(z-1)^3}$ with center $z = 1$. It is valid in the ring $1 < |z| < \infty$.

2. Let $f(z) = \frac{1}{z^2}$ then

$$f'(z) = -\frac{2}{z^3},\ f''(z) = \frac{(-2)(-3)}{z^4}, \dots, f^{(n)}(z) = \frac{(-2)(-3)\dots(-n-1)}{z^{n+2}}$$

and therefore

$$f(3) = \frac{1}{3^2}, f'(3) = -\frac{2}{3^3}, \dots, f^{(n)}(3) = \frac{(-1)^n (n+1)!}{3^{n+2}}$$

Thus the Taylor series of $f(z)$ with center $z = 3$ is given by

$$\frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)!}{3^{n+2} n!}(z-3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{3^{n+2}}(z-3)^n$$

Thus

$$\frac{1}{z^2(z-3)^2} = \sum_{n=0}^{\infty} \frac{(-1)^n (n+1)}{3^{n+2}}(z-3)^{n-2} = \frac{1}{3^2(z-3)^2} - \frac{2}{3^3(z-3)} + \sum_{m=0}^{\infty} \frac{(-1)^m (m+3)}{3^{m+4}}(z-3)^m$$

is the required Laurent series of $\frac{1}{z^2(z-3)^2}$ with center $z = 3$ valid in $0 < |z| < 3$.

■

Question 2(b) *Find the residues of $f(z) = e^z \csc^2 z$ at all its poles in the finite plane.*

Solution. The poles are at zeros of $\sin^2 z$, and $\sin^2 z = 0$ iff $z = n\pi, n \in \mathbb{Z}$, the set of integers. All these poles are double poles.

Residue at $z = n\pi$ of $f(z)$ is $\dfrac{1}{1!}\dfrac{d}{dz}\left(\dfrac{(z-n\pi)^2 e^z}{\sin^2 z}\right)_{z=n\pi}$. Now

$$\begin{aligned}\frac{d}{dz}\left(\frac{(z-n\pi)^2 e^z}{\sin^2 z}\right) &= \frac{\sin^2 z[(z-n\pi)^2 e^z + 2(z-n\pi)e^z] - (z-n\pi)^2 e^z 2\sin z\cos z}{\sin^4 z} \\ &= \frac{e^z (z-n\pi)}{\sin^3 z}\big((z-n\pi)\sin z + 2\sin z - 2(z-n\pi)\cos z\big)\end{aligned}$$

Using $\lim_{z\to n\pi} \frac{z-n\pi}{\sin z} = \frac{1}{\cos n\pi} = (-1)^n$, we get

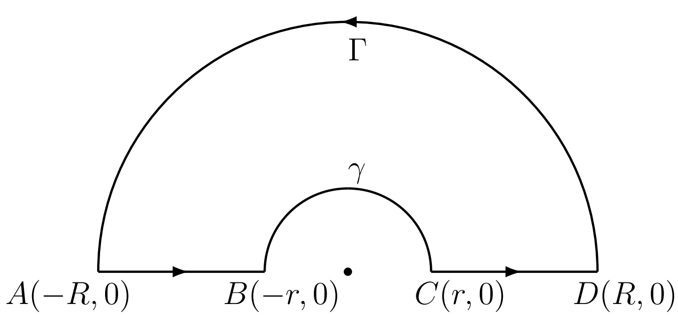
$$\begin{aligned}
\frac{d}{dz}\left(\frac{(z-n\pi)^2e^z}{\sin^2 z}\right)_{z=n\pi} &= e^{n\pi}\lim_{z\to n\pi}\frac{(z-n\pi)}{\sin^3 z}\big((z-n\pi)\sin z + 2\sin z - 2(z-n\pi)\cos z\big)\\
&= e^{n\pi}(-1)^n \lim_{z\to n\pi}\frac{(z-n\pi)(\sin z - 2\cos z) + 2\sin z}{\sin^2 z}\\
&= e^{n\pi}(-1)^n \lim_{z\to n\pi}\frac{\sin z - 2\cos z + (z-n\pi)(\cos z + 2\sin z) + 2\cos z}{2\sin z\cos z}\\
&= e^{n\pi}\lim_{z\to n\pi}\frac{\sin z + (z-n\pi)(\cos z + 2\sin z)}{2\sin z}\\
&= e^{n\pi}\lim_{z\to n\pi}\frac{\cos z + \cos z + 2\sin z + (z-n\pi)(-\sin z + 2\cos z)}{2\cos z}\\
&= e^{n\pi}
\end{aligned}$$

Thus the residue at $z = n\pi$ of $e^z \csc^2 z$ is $e^{n\pi}$. ∎

Question 2(c) *By means of contour integration evaluate* $\int_0^\infty \frac{(\log_e u)^2}{u^2+1}\,du$.

Solution.

We take $f(z) = \frac{(\log z)^2}{z^2+1}$ and the contour C consisting of the line joining $(-R, 0)$ to $(-r, 0)$, the semicircle γ of radius r with center $(0, 0)$, the line joining $(r, 0)$ to $(R, 0)$ and Γ a semicircle of radius R with center $(0, 0)$. The contour lies in the upper half plane and is oriented anticlockwise. We have avoided the branch point $z = 0$ of the multiple valued function $\log z$.



(Eventually we shall let $R \to \infty, r \to 0$).

(1) On Γ, $z = Re^{i\theta}$ and $|1 + z^2| \geq |z|^2 - 1 = R^2 - 1$. Thus

$$\begin{aligned}
\left|\int_\Gamma f(z)\,dz\right| &\leq \left|\int_0^\pi \frac{\big(\log(Re^{i\theta})\big)^2}{R^2-1} iRe^{i\theta}\,d\theta\right|\\
&\leq \int_0^\pi \frac{|\log R + i\theta|^2}{R^2-1}R\,d\theta\\
&= \frac{R}{R^2-1}\int_0^\pi ((\log R)^2 + \theta^2)\,d\theta = \frac{R}{R^2-1}\left(\pi(\log R)^2 + \frac{\pi^3}{3}\right)
\end{aligned}$$

But $\frac{R}{R^2-1}\left(\pi(\log R)^2 + \frac{\pi^3}{3}\right) \to 0$ as $R \to \infty$, therefore

$$\lim_{R\to\infty}\int_\Gamma f(z)\,dz = 0$$

(2) On γ, $z = re^{i\theta}$, $|z|^2 + 1 \geq 1 - |z|^2 = 1 - r^2$. Thus

$$\left| \int_\gamma f(z)\, dz \right| \leq \int_\pi^0 \frac{(\log r)^2 + \theta^2}{1 - r^2} r\, d\theta = \frac{r}{1 - r^2}\Big(\pi(\log r)^2 + \frac{\pi^3}{3}\Big)$$

But the right side $\to 0$ as $r \to 0$, it follows that $\lim_{r \to 0} \int_\gamma f(z)\, dz = 0$.

(3) $f(z)$ has a simple pole at $z = i$ in the the upper half plane (inside C) and the residue at $z = i$ of $f(z)$ is $\frac{(\log i)^2}{2i} = \frac{1}{2i}\Big(\frac{\pi i}{2}\Big)^2 = \frac{\pi^2 i}{8}$. Thus

$$\lim_{R \to \infty, r \to 0} \int_C f(z)\, dz = \lim_{R \to \infty, r \to 0} \int_r^R f(x)\, dx + \int_R^r f(xe^{i\pi})\, dxe^{i\pi} = 2\pi i \frac{\pi^2 i}{8}$$

because on the line CD, $z = x$, and on the line AB, $z = xe^{i\pi}$. Hence

$$-\int_\infty^0 \frac{(\log(xe^{i\pi}))^2}{1 + x^2 e^{2\pi i}}\, dx + \int_0^\infty \frac{(\log x)^2}{1 + x^2}\, dx = -\frac{\pi^3}{4}$$

Now $(\log(xe^{i\pi}))^2 = (\log x)^2 - \pi^2 + 2i\pi \log x$, so

$$2\int_0^\infty \frac{(\log x)^2}{1 + x^2}\, dx - \pi^2 \int_0^\infty \frac{dx}{1 + x^2} + 2i\pi \int_0^\infty \frac{\log x}{1 + x^2}\, dx = -\frac{\pi^3}{4}$$

Equating real parts, and noting that $\int_0^\infty \frac{dx}{1 + x^2} = \tan^{-1} x\Big]_0^\infty = \frac{\pi}{2}$, we get

$$2\int_0^\infty \frac{(\log x)^2}{1 + x^2}\, dx = \frac{\pi^3}{2} - \frac{\pi^3}{4} = \frac{\pi^3}{4}$$

so that $\int_0^\infty \frac{(\log x)^2}{1 + x^2}\, dx = \frac{\pi^3}{8}$.

Note that by equating imaginary parts, we get $\int_0^\infty \frac{\log x}{1 + x^2}\, dx = 0$. ∎