UPSC Civil Services Main 1995 - Mathematics
Complex Analysis

Brij Bhooshan

Asst. Professor

B.S.A. College of Engg & Technology

Mathura

Question 1(a) *Let $u(x, y) = 3x^2y + 2x^2 - y^3 - 2y^2$. Prove that u is a harmonic function. Find a harmonic function v such that $u + iv$ is an analytic function of z.*

Solution. Clearly

$$\frac{\partial u}{\partial x} = 6xy + 4x \quad , \quad \frac{\partial u}{\partial y} = 3x^2 - 3y^2 - 4y$$
$$\frac{\partial^2 u}{\partial x^2} = 6y + 4 \quad , \quad \frac{\partial^2 u}{\partial y^2} = -6y - 4$$

Thus $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, showing that u is a harmonic function.

Let $f(z) = u + iv$, where v is to be so determined that $f(z)$ is analytic and v is harmonic. Such a function v along with u would have to satisfy the Cauchy-Riemann equations $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$. Now

$$\begin{aligned} f'(z) &= \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} \\ &= 6xy + 4x - i(3x^2 - 3y^2 - 4y) \\ &= -3i(x^2 - y^2 + 2ixy) + 4(x + iy) \\ &= -3iz^2 + 4z \end{aligned}$$

Thus

$$\begin{aligned} f(z) &= 2z^2 - iz^3 \\ &= 2(x+iy)^2 - i(x+iy)^3 \\ &= 2x^2 - 2y^2 + 4ixy - ix^3 + 3x^2y + 3ixy^2 - y^3 \\ &= 3x^2y + 2x^2 - y^3 - 2y^2 + i(4xy - x^3 + 3xy^2) \end{aligned}$$

Thus $v = 4xy - x^3 + 3xy^2$. Clearly

$$\frac{\partial v}{\partial x} = 4y - 3x^2 + 3y^2 \quad , \quad \frac{\partial v}{\partial y} = 4x + 6xy$$
$$\frac{\partial^2 v}{\partial x^2} = -6x \quad , \quad \frac{\partial^2 v}{\partial y^2} = 6x$$

so that $\dfrac{\partial^2 v}{\partial x^2} + \dfrac{\partial^2 v}{\partial y^2} = 0$, showing that v is a harmonic function. ∎

Question 1(b) *Find the Taylor series expansion of $f(z) = \dfrac{z}{z^4+9}$ around $z = 0$. Find also the radius of convergence.*

Solution. It is obvious that

$$\begin{aligned} f(z) &= \frac{z}{9}\Big(1 + \frac{z^4}{9}\Big)^{-1} = \frac{z}{9}\Big(1 - \frac{z^4}{9} + \frac{z^8}{81} - \frac{z^{12}}{729} + \dots\Big) \\ &= \frac{z}{9}\sum_{n=0}^{\infty}(-1)^n\Big(\frac{z^4}{9}\Big)^n = \sum_{n=0}^{\infty}(-1)^n\frac{z^{4n+1}}{9^{n+1}} \end{aligned}$$

provided $|\frac{z^4}{9}| < 1$. This indeed is Taylor's series representation of $f(z)$ which to start with is valid for $|\frac{z^4}{9}| < 1$. The radius of convergence of a power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ is given by $\Big(\limsup |a_n|^{\frac{1}{n}}\Big)^{-1}$. In this case the radius of convergence is $\Big(\displaystyle\lim_{n\to\infty}\Big(\frac{1}{9^{n+1}}\Big)^{\frac{1}{4n+1}}\Big)^{-1} = \displaystyle\lim_{n\to\infty} 9^{\frac{n+1}{4n+1}} = 9^{\frac{1}{4}} = \sqrt{3}$.

Note: We did not get the radius of convergence greater than the disc of validity namely $|\frac{z^4}{9}| < 1$ as we have a singularity of $f(z)$ on $|z| = \sqrt{3}$, namely those z for which $z^4 = -9 = i^2 9$ or $z^2 = \pm 3i$. ∎

Question 1(c) *Let C be a circle $|z| = 2$ oriented counter-clockwise. Evaluate the integral $\displaystyle\int_C \frac{\cosh \pi z}{z(z^2+1)}\, dz$ with the aid of residues.*

Solution. By Cauchy's residue theorem, $\displaystyle\int_C \frac{\cosh \pi z}{z(z^2+1)}\,dz = 2\pi i($sum of residues at poles of $\displaystyle\frac{\cosh \pi z}{z(z^2+1)}$ inside C).

The only poles of $\displaystyle\frac{\cosh \pi z}{z(z^2+1)}$ are at $z = 0, \pm i$ all within $|z| = 2$. All these are simple poles.

Residue at $z = 0$ is $\displaystyle\lim_{z\to 0} \frac{z\cosh \pi z}{z(z^2+1)} = 1$.

Residue at $z = i$ is $\displaystyle\lim_{z\to i} \frac{(z-i)\cosh \pi z}{z(z^2+1)} = \frac{\cosh \pi i}{i\cdot 2i} = -\frac{\cos \pi}{2} = \frac{1}{2}$.

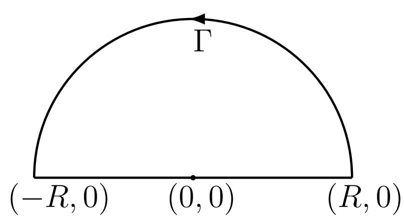
Residue at $z = -i$ is $\displaystyle\lim_{z\to -i} \frac{(z+i)\cosh \pi z}{z(z^2+1)} = \frac{\cosh(-\pi i)}{(-i)\cdot(-2i)} = \frac{1}{2}$.

Thus $\displaystyle\int_C \frac{\cosh \pi z}{z(z^2+1)}\,dz = 2\pi i\Big[1 + \frac{1}{2} + \frac{1}{2}\Big] = 4\pi i$. ∎

Question 2(a) *Evaluate the integral* $\displaystyle\int_0^\infty \frac{\cos ax}{x^2+1}\,dx,\ a \geq 0$.

Solution.

Let $f(z) = \dfrac{e^{iaz}}{z^2+1}$. Let γ be the contour consisting of the line joining $(-R, 0)$ and $(R, 0)$ and Γ, which is the arc of the circle of radius R and center $(0, 0)$ lying in the upper half plane. γ is oriented counter-clockwise.



$$\lim_{R\to\infty} \int_\gamma f(z)\,dz = \int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx + \lim_{R\to\infty} \int_\Gamma \frac{e^{iaz}}{z^2+1}\,dz$$

Since $|z^2+1| \geq R^2 - 1$ on Γ and $|e^{iaz}| = |e^{iaRe^{i\theta}}| = |e^{-aR\sin\theta}| \leq 1$ because $\sin\theta \geq 0$ in $0 \leq \theta \leq \pi$, so

$$\left|\int_\Gamma \frac{e^{iaz}}{z^2+1}\,dz\right| \leq \frac{\pi R}{R^2-1}$$

as $dz = iRe^{i\theta}\,d\theta$, showing that $\displaystyle\lim_{R\to\infty} \int_\Gamma \frac{e^{iaz}}{z^2+1}\,dz = 0$.

By Cauchy's residue theorem, $\lim_{R\to\infty} \int_\gamma f(z)\,dz = 2\pi i($ Sum of residues at poles of $\dfrac{e^{iaz}}{z^2+1}$ in the upper half plane). $z = i$ is the only pole of $f(z)$ in the upper half plane, and the residue there is given by $\displaystyle\lim_{z\to i} \frac{(z-i)e^{iaz}}{z^2+1} = \frac{e^{-a}}{2i}$.

Thus $\displaystyle\int_{-\infty}^{\infty} \frac{e^{iax}}{x^2+1}\,dx = \pi e^{-a}$, so

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a}, \quad \int_{-\infty}^{\infty} \frac{\sin ax}{x^2+1}\,dx = 0$$

Since $\dfrac{\cos ax}{x^2+1}$ is an even function of x,

$$\int_0^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos ax}{x^2+1}\,dx = \frac{\pi e^{-a}}{2}$$

∎

Question 2(b) *Let f be analytic in the entire complex plane. Suppose that there exists a constant $A > 0$, such that $|f(z)| \leq A|z|$ for all z. Prove that there is a complex number a such that $f(z) = az$ for all z.*

Solution. We first prove (Cauchy's inequality) that if $f(z)$ is analytic in a domain G and if the disc $|z - z_0| \leq \rho \subseteq G$ then

$$|f^{(n)}(z_0)| \leq \frac{n!M(\rho)}{\rho^n}$$

where $M(\rho) = \max|f(z)|$ on $|z - z_0| = \rho$ — this follows from Cauchy's Integral formula:

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{|z-z_0|=\rho} \frac{f(z)}{(z-z_0)^{n+1}}\,dz$$

and therefore

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi}\,\frac{M(\rho)}{\rho^{n+1}}\,2\pi\rho = \frac{n!M(\rho)}{\rho^n}$$

We now prove that if $f(z)$ is entire i.e. analytic over the whole complex plane, and $|f(z)| \leq G|z|^m$ for all $|z| > R$, then $f(z)$ is a polynomial of degree $\leq m$.

Let $f(z) = \sum_{n=0}^\infty a_n z^n$ be a Taylor series of $f(z)$ around $z = 0$. Then $a_n = \dfrac{f^{(n)}(0)}{n!}$. By Cauchy's inequality proved above, $|a_n| = \left|\dfrac{f^{(n)}(0)}{n!}\right| \leq \dfrac{M(r)}{r^n}$ where $M(r)$ is maximum of $|f(z)|$ on $|z| = r$. Let $r > R$, then $M(r) \leq Gr^m$ and we get $|a_n| \leq \dfrac{Gr^m}{r^n} = \dfrac{G}{r^{n-m}}$. and therefore as $r \to \infty$, $\dfrac{G}{r^{n-m}} \to 0$ for $n > m$ i.e. $|a_n| = 0$ for $n > m$. Hence $f(z) = \sum_{r=0}^m a_r z^r$ i.e. $f(z)$ is a polynomial of degree $\leq m$.

Now we are given $|f(z)| \leq A|z|$. This means that $f(z) = a_0 + a_1 z$. But $0 \leq |f(0)| \leq A \cdot 0 \Rightarrow f(0) = 0 \Rightarrow a_0 = 0$, so $f(z) = a_1 z$, where a_1 is a constant.

Note: An alternative statement of the above question is: If $f(z)$ is an entire transcendental function, then whatever $G > 0, R > 0, m > 0$ are prescribed, there exist points z such that $|f(z)| > G|z|^m$ and $|z| > R$. ∎

Alternate solution: Consider the function $g(z) = \frac{f(z)}{z}, z \neq 0$ and $g(0) = f'(0)$. Note that $|f(z)| \leq A|z| \Rightarrow f(0) = 0$. Then g is continuous at 0, because

$$\lim_{z\to 0} |g(z) - g(0)| = \lim_{z\to 0}\left|\frac{f(z)}{z} - f'(0)\right| = \lim_{z\to 0}\left|\frac{f(z) - f(0)}{z} - f'(0)\right| = 0$$

Let $f = u + iv$, where u, v satisfy the Cauchy Riemann equations, since f is entire. Then

$$g(z) = \frac{u + iv}{x + iy} = \frac{(ux + yv) + i(vx - uy)}{x^2 + y^2}$$

Writing $g(z) = U + iV$, we get $U = \dfrac{ux + yv}{x^2 + y^2}, V = \dfrac{vx - uy}{x^2 + y^2}$. Now it is clear that g is analytic over the entire complex plane except possibly at $z = 0$. We now check the Cauchy Riemann equations for U, V at $z = 0$. Note that $f(0) = 0 \Rightarrow u(0,0) = v(0,0) = 0$.

$$\begin{aligned}
\frac{\partial U}{\partial x}(0,0) &= \lim_{h\to 0}\frac{U(h,0) - U(0,0)}{h} = \lim_{h\to 0}\frac{\frac{u(h,0)}{h} - u_x(0,0)}{h} = \lim_{h\to 0}\frac{u(h,0) - hu_x(0,0)}{h^2} \\
&= \lim_{h\to 0}\frac{u_x(h,0) - u_x(0,0)}{2h} = \frac{1}{2}u_{xx}(0,0) \\
\frac{\partial U}{\partial y}(0,0) &= \lim_{k\to 0}\frac{U(0,k) - U(0,0)}{k} = \lim_{k\to 0}\frac{\frac{v(0,k)}{k} - u_x(0,0)}{k} = \lim_{k\to 0}\frac{v(0,k) - ku_x(0,0)}{k^2} \\
&= \lim_{k\to 0}\frac{v_y(0,k) - u_x(0,0)}{2k} = \frac{1}{2}v_{yy}(0,0) \\
\frac{\partial V}{\partial x}(0,0) &= \lim_{h\to 0}\frac{V(h,0) - V(0,0)}{h} = \lim_{h\to 0}\frac{\frac{v(h,0)}{h} - v_x(0,0)}{h} = \lim_{h\to 0}\frac{v(h,0) - hv_x(0,0)}{h^2} \\
&= \lim_{h\to 0}\frac{v_x(h,0) - v_x(0,0)}{2h} = \frac{1}{2}v_{xx}(0,0) \\
\frac{\partial V}{\partial y}(0,0) &= \lim_{k\to 0}\frac{V(0,k) - V(0,0)}{k} = \lim_{k\to 0}\frac{\frac{-u(0,k)}{k} - v_x(0,0)}{k} = \lim_{k\to 0}\frac{-u(0,k) - kv_x(0,0)}{k^2} \\
&= \lim_{k\to 0}\frac{-u_y(0,k) - v_x(0,0)}{2k} = -\frac{1}{2}u_{yy}(0,0)
\end{aligned}$$

Now by the Cauchy Riemann equations for u, v, $u_x = v_y \Rightarrow u_{xx} = v_{xy}$ and $u_y = -v_x \Rightarrow u_{yy} = -v_{yx}$. Hence $U_x(0,0) = \frac{1}{2}u_{xx}(0,0) = \frac{1}{2}v_{xy}(0,0) = -\frac{1}{2}u_{yy}(0,0) = V_y(0,0)$.

Also, $v_x = -u_y \Rightarrow v_{xx} = -u_{xy}$, and $v_y = u_x \Rightarrow v_{yy} = u_{yx}$. So $U_y(0,0) = \frac{1}{2}v_{yy}(0,0) = \frac{1}{2}u_{yx} = -\frac{1}{2}v_{xx}(0,0) = -V_x(0,0)$. Thus the Cauchy Riemann equations hold at $(0,0)$ also, so $g(z)$ is analytic at 0, as it is continuous at 0. Thus $g(z)$ is an entire function.

But $|g(z)| = |\frac{f(z)}{z}| \leq \frac{A|z|}{|z|} = A$, so g is bounded over the complex plane. Hence by Liouville's theorem, g is a constant, say a. Thus $f(z) = az$, as required.

Question 2(c) *Suppose a power series* $\displaystyle\sum_{n=0}^{\infty} a_n z^n$ *converges at a point* $z_0 \neq 0$. *Let* z_1 *be such that* $|z_1| < |z_0|$ *and* $z_1 \neq 0$. *Show that the series converges uniformly in the disc* $\{z \,:\, |z| \leq |z_1|\}$.

Solution. Let $|\frac{z_1}{z_0}| = \rho$, then $\rho < 1$. Since $\sum_{n=0}^{\infty} a_n z_0^n$ is convergent, $a_n z_0^n \to 0$ as $n \to \infty$, therefore there exists M such that $|a_n z_0^n| < M$ for $n \geq 0$. Now let z be any point such that $|z| \leq |z_1|$, then

$$\Big|\sum_{n=r}^{r+p} a_n z^n\Big| \leq \sum_{n=r}^{r+p} |a_n z^n| = \sum_{n=r}^{r+p} \Big|a_n z_0^n \Big(\frac{z}{z_0}\Big)^n\Big| \leq M \sum_{n=r}^{r+p} \Big|\frac{z}{z_0}\Big|^n = M \sum_{n=r}^{r+p} \rho^n$$

Since the series $\sum_{n=0}^{\infty} \rho^n$ is convergent, given $\epsilon > 0$ there exists N such that $\sum_{n=r}^{r+p} \rho^n < \frac{\epsilon}{M}$ for all $r \geq N$ and $p = 1, 2, \ldots$. Clearly this N is independent of z. Thus given $\epsilon > 0$ there exists N independent of z such that

$$\Big|\sum_{n=r}^{r+p} a_n z^n\Big| < \epsilon \ \text{ for } n \geq N, p = 1, 2, 3, \ldots$$

i.e. the series $\sum_{n=0}^{\infty} a_n z^n$ is uniformly convergent for all z with $|z| \leq |z_1|$. ∎