UPSC Civil Services Main 1996 - Mathematics
Complex Analysis

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Question 1(a) *Sketch the ellipse C described in the complex plane by*

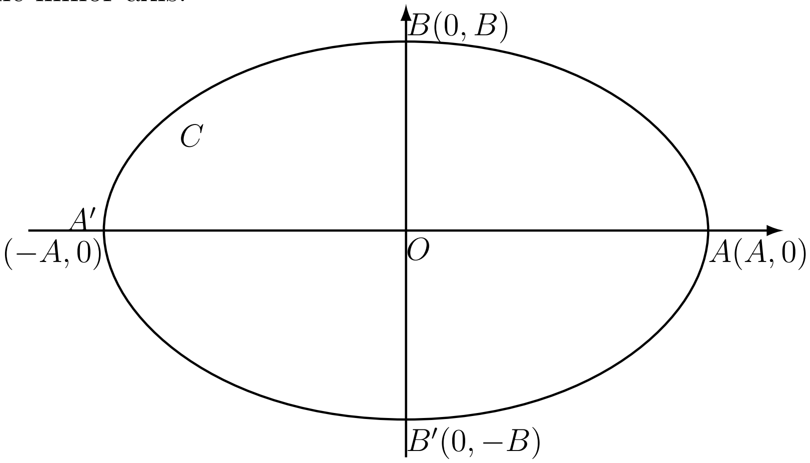
$$z = A\cos\lambda t + iB\sin\lambda t, A > B$$

where t is a real variable and A, B, λ are positive constants.

If C is the trajectory of a particle with $z(t)$ as the position vector of the particle at time t, identify with justification

1. *the two positions where the velocity is minimum.*
2. *the two positions where the acceleration is maximum.*

Solution. We are given that $x = A\cos\lambda t, y = B\sin\lambda t$ which implies that $\dfrac{x^2}{A^2} + \dfrac{y^2}{B^2} = 1$. Since $A > B$, it follows that it is the standard ellipse with $2A$ as the major axis and $2B$ as the minor axis.



1. The velocity $v = \dfrac{dz}{dt} = -A\lambda \sin \lambda t + iB\lambda \cos \lambda t$.

$$\begin{aligned} \text{Speed} &= \text{magnitude of velocity} = \left|\frac{dz}{dt}\right| \\ &= \sqrt{A^2\lambda^2 \sin^2 \lambda t + B^2\lambda^2 \cos^2 \lambda t} \\ &= \lambda\sqrt{(A^2 - B^2)\sin^2 \lambda t + B^2} \end{aligned}$$

Since $A^2 - B^2 > 0$, the speed is minimum when $\sin^2 \lambda t = 0$ i.e. when $x(t) = \pm A, y(t) = 0$ i.e. when the particle is at the two ends of the major axis, the points A and A' in the figure.

2. Acceleration $= \dfrac{d^2z}{dt^2} = -A\lambda^2 \cos \lambda t - iB\lambda^2 \sin \lambda t$.

Magnitude of acceleration $= \lambda^2\sqrt{A^2 \cos^2 \lambda t + B^2 \sin^2 \lambda t} = \lambda^2\sqrt{(A^2 - B^2)\cos^2 \lambda t + B^2}$. Since $A^2 - B^2 > 0$, acceleration is maximum when $\cos^2 \lambda t = 1 \Rightarrow \cos \lambda t = \pm 1$ i.e. the particle is at either end of the major axis, A or A'. (Note that acceleration is minimum when $\cos^2 \lambda t = 0$ i.e. the particle is at either end of the minor axis).

■

Question 1(b) *Evaluate* $\displaystyle\lim_{z\to 0} \frac{1 - \cos z}{\sin(z^2)}$.

Solution.

$$\lim_{z\to 0} \frac{1 - \cos z}{\sin(z^2)} = \lim_{z\to 0} \frac{2\sin^2 \frac{z}{2}}{\sin(z^2)} = \lim_{z\to 0} \frac{2}{4} \frac{\frac{\sin^2 \frac{z}{2}}{(\frac{z}{2})^2}}{\frac{\sin(z^2)}{z^2}} = \frac{1}{2}$$

Note that $\sin z$ has a simple zero at $z = 0$ and $\sin z = z\phi(z)$ where $\phi(z)$ is analytic and $\phi(0) = 1$, so $\displaystyle\lim_{z\to 0} \frac{\sin z}{z} = 1$. ■

Question 1(c) *Show that $z = 0$ is not a branch point for the function $f(z) = \dfrac{\sin\sqrt{z}}{\sqrt{z}}$. Is it a removable singularity?*

Solution. We know that $w = \sqrt{z}$ is a multiple valued function and has two branches. Once we fix a branch of $w = \sqrt{z}$, $\sin\sqrt{z}$ is analytic, and

$$\sin\sqrt{z} = \sqrt{z} - \frac{(\sqrt{z})^3}{3!} + \frac{(\sqrt{z})^5}{5!} + \dots$$

or

$$\frac{\sin\sqrt{z}}{\sqrt{z}} = 1 - \frac{z}{3!} + \frac{z^2}{5!} - \frac{z^3}{7!} + \dots$$

Thus $\lim_{z\to 0} \frac{\sin\sqrt{z}}{\sqrt{z}} = 1$, so $z = 0$ is not a branch point of the function $f(z) = \frac{\sin\sqrt{z}}{\sqrt{z}}$. In fact $z = 0$ is a removable singularity of $f(z)$. In fact

$$F(z) = \begin{cases} \frac{\sin\sqrt{z}}{\sqrt{z}}, & z \neq 0 \\ 1, & z = 0 \end{cases}$$

is analytic everywhere once a branch of $\sqrt{z}$ is specified. ∎

Question 2(a) *Prove that every polynomial equation* $a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n = 0, a_n \neq 0, n \geq 1$ *has exactly* n *roots.*

Solution. Let $P(z) = a_0 + a_1 z + a_2 z^2 + \ldots + a_n z^n$. Suppose, if possible, that $P(z) \neq 0$ for any $z \in \mathbb{C}$. Let $f(z) = \frac{1}{P(z)}$, then $f(z)$ is an entire function i.e. $f(z)$ is analytic in the whole complex plane. We shall now show that $f(z)$ **is bounded**.

$$P(z) = z^n \left(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_0}{z^n} \right)$$

Since $\frac{a_j}{z^{n-j}} \to 0$ as $z \to \infty$, for $0 \leq j < n$, is follows that given $\epsilon = \frac{|a_n|}{2n}$ there exists $R > 0$ such that $|z| > R \Rightarrow \left| \frac{a_j}{z^{n-j}} \right| < \frac{|a_n|}{2n}$ for $0 \leq j < n$. Thus

$$\left| a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_0}{z^n} \right| \geq |a_n| - n\left| \frac{a_n}{2n} \right| = \left| \frac{a_n}{2} \right|$$

and therefore

$$|f(z)| = \left| \frac{1}{P(z)} \right| = \left| \frac{1}{z^n \left(a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_0}{z^n} \right)} \right| \leq \frac{2}{|a_n| R^n} \quad \text{for } |z| > R$$

Since $|z| \leq R$ is a compact set and $f(z)$ is analytic on it, $f(z)$ is bounded on $|z| \leq R$. Consequently $f(z)$ is bounded on the whole complex plane. Now we use Liouville's theorem — If an entire function is bounded on the whole complex plane, then it is a constant. Thus $f(z)$ and therefore $P(z)$ is a constant, which is not true, hence our assumption that $P(z) \neq 0$ for all $z \in \mathbb{C}$ is false. So there is at least one $z_1 \in \mathbb{C}$ where $P(z_1) = 0$. (This result is called the fundamental theorem of algebra.)

We now prove by induction on n that $P(z)$ has n zeros. If $n = 1$, $P(z) = a_0 + a_1 z$ has one zero namely $z = -\frac{a_0}{a_1}$.

Assume as induction hypothesis that any polynomial of degree $n-1$ has $n-1$ zeros. By Euclid's algorithm, , we get $P_1(z)$ and $R(z)$ such that $P(z) = (z - z_1)P_1(z) + R(z)$, where $R(z) \equiv 0$ or $\deg R(z) < 1$ i.e. $R(z)$ is a constant. Putting $z = z_1$ we get $R(z) \equiv 0$, so $P(z) = (z - z_1)P_1(z)$. Since $P_1(z)$ is a polynomial of degree $n - 1$, by induction hypothesis it has $n - 1$ roots in $\mathbb{C}$, and therefore $P(z)$ has n roots in $\mathbb{C}$.

We now prove that $P(z)$ has exactly n roots. Let $z_1, z_2, \ldots, z_n$ be the (not necessarily distinct) roots of $P(z)$. Let $g(z) = \dfrac{P(z)}{(z-z_1)(z-z_2)\ldots(z-z_n)}$. Clearly $g(z)$ is analytic in the whole complex plane. Since

$$\lim_{z\to\infty} g(z) = \lim_{z\to\infty} \frac{P(z)}{(z-z_1)(z-z_2)\ldots(z-z_n)} = \frac{a_n + \frac{a_{n-1}}{z} + \frac{a_{n-2}}{z^2} + \ldots + \frac{a_0}{z^n}}{(1-\frac{z_1}{z})(1-\frac{z_2}{z})\ldots(1-\frac{z_n}{z})} = a_n$$

it follows that given $\epsilon > 0$ there exists R such that $|g(z) - a_n| < \epsilon$ for $|z| > R$, so $g(z)$ is bounded in the region $|z| > R$. The function $g(z)$ being analytic is bounded in the compact region $|z| \le R$. Thus by Liouville's theorem $g(z)$ is a constant, in fact $g(z) = a_n$, and therefore

$$P(z) = a_n(z-z_1)(z-z_2)\ldots(z-z_n)$$

Thus if ζ is a zero of $P(z)$, then $\zeta = z_j$ for some j, $1 \le j \le n$. Thus $P(z)$ has exactly n zeroes. ∎

Alternate Proof: We shall use Rouche's theorem — Let γ be a simple closed rectifiable curve. Let $f(z), g(z)$ be analytic on and within γ. Suppose $|g(z)| < |f(z)|$ on γ, then $f(z)$ and $f(z) \pm g(z)$ have the same number of zeroes inside γ.

Let $f(z) = a_n z^n$ and $g(z) = a_{n-1}z^{n-1} + \ldots + a_0$. Let R be so large that $|g(z)| < |f(z)|$ on $|z| = R$. Then $f(z)$ and $f(z) + g(z) = P(z)$ have the same number of zeroes within $|z| = R$. But whatever $R > 0$ we take, $f(z)$ has exactly n zeroes in $|z| = R$, therefore $P(z)$ has exactly n zeroes in $\mathbb{C}$.

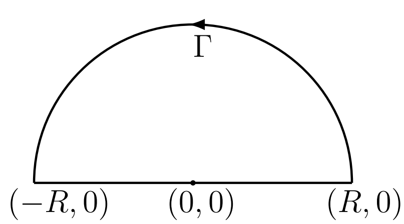
Note: Rouche's theorem follows from the Argument Principle — Note that $\Delta_\gamma(\arg(f(z)+g(z)))$ =change in argument of $f(z)+g(z)$ as z moves along $\gamma = \Delta_\gamma \arg f(z) + \Delta_\gamma \arg(1+\frac{g(z)}{f(z)})$ as $f(z) \ne 0$ along γ. But $\Delta_\gamma \arg(1+\frac{g(z)}{f(z)}) = 0$ because $|\frac{g(z)}{f(z)}| < 1$ and therefore $\frac{g(z)}{f(z)}$ continues to lie in the disc $|w-1| < 1$ as z moves on γ i.e. does not go around the origin.

Question 2(b) *By using the residue theorem, evaluate*

$$\int_0^\infty \frac{\log_e(x^2+1)}{x^2+1}\,dx$$

Solution.

Let $f(z) = \dfrac{\log(z+i)}{1+z^2}$ and we consider $\log(z+i)$ in $\mathbb{C} - \{z \mid z = iy, y \le -1\}$, where it is single-valued. Let γ be the contour consisting of the line joining $(-R,0)$ and $(R,0)$ and Γ, which is the arc of the circle of radius R and center $(0,0)$ lying in the upper half plane. γ is oriented counter-clockwise.



Clearly $f(z)$ has a simple pole at $z = i$ in the upper half plane. The residue at $z = i$ is

$$\lim_{z\to i} \frac{(z+i)\log(z+i)}{1+z^2} = \frac{\log 2i}{2i} = \frac{1}{2i}\log 2e^{\frac{\pi i}{2}} = \frac{1}{2i}\Big[\log 2 + i\frac{\pi}{2}\Big] = \frac{\pi}{4} - \frac{1}{2}i\log 2$$

Thus by Cauchy's residue theorem

$$\lim_{R\to\infty}\int_\gamma \frac{\log(z+i)}{1+z^2} = \lim_{R\to\infty}\int_\Gamma \frac{\log(z+i)}{1+z^2} + \int_{-\infty}^{\infty}\frac{\log(x+i)}{1+x^2}\,dx = 2\pi i\Big[\frac{\pi}{4} - \frac{1}{2}i\log 2\Big]$$

as $z = x$ on the real axis.

We shall now show that $\lim_{R\to\infty}\int_\Gamma \frac{\log(z+i)}{1+z^2} = 0$. On Γ, $z = Re^{i\theta}$, so

$$\left|\int_\Gamma \frac{\log(z+i)}{1+z^2}\right| = \left|\int_0^\pi \frac{\log(Re^{i\theta}+i)Rie^{i\theta}}{R^2e^{2i\theta}+1}\,d\theta\right|$$

Now $|R^2e^{2i\theta}+1| \geq R^2-1, \log(Re^{i\theta}+i) = \log Re^{i\theta} + \log(1+\frac{i}{Re^{i\theta}})$. Clearly $|\log Re^{i\theta}| = |\log R + i\theta| \leq \log R + \pi$ and therefore

$$\left|\int_\Gamma \frac{\log(z+i)}{1+z^2}\right| \leq \int_0^\pi \frac{(\pi+\log R)R}{R^2-1}\,d\theta + \int_0^\pi \frac{R|\log(1+\frac{i}{Re^{i\theta}})|}{R^2-1}\,d\theta$$

Since $\frac{(\pi+\log R)R}{R^2-1} \to 0$ and $\frac{R|\log(1+\frac{i}{Re^{i\theta}})|}{R^2-1} \to 0$ as $R\to\infty$, it follows that $\lim_{R\to\infty}\int_\Gamma \frac{\log(z+i)}{1+z^2} = 0$.

Thus

$$\int_{-\infty}^{\infty}\frac{\log(x+i)}{1+x^2}\,dx = \pi\log 2 + i\frac{\pi^2}{2}$$

Equating real and imaginary parts, we get

$$\int_0^\infty \frac{\log(1+x^2)}{1+x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\log(1+x^2)}{1+x^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\log(x+i)+\log(x-i)}{1+x^2}\,dx = \frac{1}{2}[2\pi\log 2] = \pi\log 2$$

∎

Question 2(c) *Find the Laurent expansion of* $f(z) = (z-3)\sin\Big(\frac{1}{z+2}\Big)$ *about the singularity* $z = -2$. *Specify the region of convergence and the nature of the singularity at* $z = -2$.

Solution. It is well known that

$$\begin{aligned}
\sin\Big(\frac{1}{z+2}\Big) &= \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)!}\Big(\frac{1}{z+2}\Big)^{2k-1} \\
\Rightarrow (z-3)\sin\Big(\frac{1}{z+2}\Big) &= (z+2)\sin\Big(\frac{1}{z+2}\Big) - 5\sin\Big(\frac{1}{z+2}\Big) \\
&= (z+2)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)!}\Big(\frac{1}{z+2}\Big)^{2k-1} - 5\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{(2k-1)!}\Big(\frac{1}{z+2}\Big)^{2k-1} \\
&= \sum_{k=0}^{\infty}\frac{a_k}{(z+2)^k}, \quad a_{2k-2} = \frac{(-1)^{k-1}}{(2k-1)!}, \quad a_{2k-1} = \frac{5(-1)^{k-1}}{(2k-1)!}
\end{aligned}$$

The region of convergence of the series is $0 < |z+2| < \infty$. The Laurent expansion shows that the function has an essential singularity at $z = -2$ — this also follows from the fact that $\lim_{z\to 0} \sin \frac{1}{z}$ does not exist. ∎