# UPSC Civil Services Main 1997 - Mathematics Complex Analysis

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**Question 1(a)** *Prove that* $u = e^x(x\cos y - y\sin y)$ *is harmonic and find the analytic function whose real part is* $u$*.*

**Solution.**

$$\begin{aligned}
\frac{\partial u}{\partial x} &= e^x(x\cos y - y\sin y) + e^x\cos y \\
\frac{\partial^2 u}{\partial x^2} &= e^x(x\cos y - y\sin y) + 2e^x\cos y \\
\frac{\partial u}{\partial y} &= e^x(-x\sin y - \sin y - y\cos y) \\
\frac{\partial^2 u}{\partial y^2} &= e^x(-x\cos y - 2\cos y + y\sin y)
\end{aligned}$$

Clearly $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, showing that $u$ is harmonic.

Let $f(z) = u\Big(\dfrac{z+\overline{z}}{2}, \dfrac{z-\overline{z}}{2i}\Big) + iv\Big(\dfrac{z+\overline{z}}{2}, \dfrac{z-\overline{z}}{2i}\Big)$. Then

$$\begin{aligned}
f'(z) &= u_x + iv_x = u_x - iu_y \text{ because of the C-R equations} \\
&= u_x(x,y) - iu_y(x,y) = u_x\Big(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\Big) - iu_y\Big(\frac{z+\overline{z}}{2}, \frac{z-\overline{z}}{2i}\Big)
\end{aligned}$$

Since the above is an identity, we take $z = \overline{z}$, so $x = z, y = 0$. Thus $f'(z) = u_x(z,0) - iu_y(z,0) = ze^z + e^z$. Then

$$f(z) = \int f'(z)\,dz = \int (z+1)e^z\,dz = ze^z + C$$

Hence $f(z) = ze^z$ is the required function. ∎

**Question 1(b)** *Evaluate* $\oint_C \frac{dz}{z+2}$ *where* $C$ *is the unit circle. Deduce that* $\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = 0$.

**Solution.** Cauchy's theorem implies that $\oint_C \frac{dz}{z+2} = 0$ because $\frac{1}{z+2}$ has no pole inside $|z| = 1$.

Putting $z = e^{i\theta}$, we get

$$\begin{aligned}
I &= \int_{|z|=1} \frac{dz}{z+2} = \int_0^{2\pi} \frac{(i\cos\theta - \sin\theta)\,d\theta}{\cos\theta + 2 + i\sin\theta} \\
&= \int_0^{2\pi} \frac{(i\cos\theta - \sin\theta)(\cos\theta + 2 - i\sin\theta)}{(\cos\theta+2)^2 + \sin^2\theta}\,d\theta \\
&= \int_0^{2\pi} \frac{i(\cos^2\theta + \sin^2\theta + 2\cos\theta) - 2\sin\theta}{\cos^2\theta + 4\cos\theta + 4 + \sin^2\theta}\,d\theta \\
&= -\int_0^{2\pi} \frac{2\sin\theta}{5+4\cos\theta}\,d\theta + i\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta
\end{aligned}$$

Since $I = 0$, it follows that

$$\int_0^{2\pi} \frac{2\sin\theta}{5+4\cos\theta}\,d\theta = 0, \int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta}\,d\theta = 0$$

∎

**Question 1(c)** *If* $f(z) = \frac{A_1}{z-a} + \frac{A_2}{(z-a)^2} + \ldots + \frac{A_n}{(z-a)^n}$, *find the residue at* $a$ *for* $\frac{f(z)}{z-b}$ *where* $A_1, A_2, \ldots A_n, a, b$ *are constants. What is the residue at infinity?*

**Solution.** Case (1): $a \neq b$.

$$\frac{f(z)}{z-b} = \frac{A_1(z-a)^{n-1} + A_2(z-a)^{n-2} + \ldots + A_n}{(z-b)(z-a)^n}$$

showing that $\frac{f(z)}{z-b}$ has a pole of order $n$ at $z = a$. The residue at $z = a$ is the coefficient of $\frac{1}{z-a}$ in the Laurent expansion of $\frac{f(z)}{z-b}$ around $a$. Now

$$\begin{aligned}
\frac{1}{z-b} &= (z-a+a-b)^{-1} = \frac{1}{a-b}\Big(1 + \frac{z-a}{a-b}\Big)^{-1} \\
\frac{f(z)}{z-b} &= \Big[\frac{A_1}{z-a} + \frac{A_2}{(z-a)^2} + \ldots + \frac{A_n}{(z-a)^n}\Big]\frac{1}{a-b}\Big[1 - \frac{z-a}{a-b} + \Big(\frac{z-a}{a-b}\Big)^2 - \Big(\frac{z-a}{a-b}\Big)^3 + \ldots\Big]
\end{aligned}$$

Thus the coefficient of $\frac{1}{z-a}$ i.e. the residue of $\frac{f(z)}{z-b}$ is given by

$$\frac{A_1}{a-b} - \frac{A_2}{(a-b)^2} + \frac{A_3}{(a-b)^3} + \ldots + \frac{(-1)^{n-1}A_n}{(a-b)^n} = -f(b)$$

Note: The same residue could be computed by using the formula — Residue at $z = a$ is $\frac{1}{(n-1)!}\frac{d^{n-1}}{dz^{n-1}}\left((z-a)^n\frac{f(z)}{z-b}\right)$ but this calculation would be much more complicated.

Case (2): $a = b$. In this case $\frac{f(z)}{z-b}$ has a pole of order $n+1$ at $z = a$. Residue at $z = a$ is given by $\frac{1}{n!}\frac{d^n}{dz^n}[(z-a)^n f(z)] = 0$.

Residue at $\infty$:

$$\begin{aligned} f(z) &= \frac{A_1}{z}\left(1-\frac{a}{z}\right)^{-1} + \frac{A_2}{z^2}\left(1-\frac{a}{z}\right)^{-2} + \ldots + \frac{A_n}{z^n}\left(1-\frac{a}{z}\right)^{-n} \\ \frac{f(z)}{z-b} &= \frac{f(z)}{z}\left(1-\frac{b}{z}\right)^{-1} = \left[\frac{A_1}{z} + \frac{1}{z^2}(\ldots) + \frac{1}{z^3}(\ldots) + \ldots\right]\left[\frac{1}{z} + \frac{b}{z^2} + \ldots\right] \end{aligned}$$

Since the term $\frac{1}{z}$ is not present in the Laurent expansion of $\frac{f(z)}{(z-b)}$ the residue at $\infty$ is 0. ∎

**Question 2(a)** *Find the Laurent series for the function* $e^{\frac{1}{z}}$ *in* $0 < |z| < \infty$. *Deduce that*

$$\frac{1}{\pi}\int_0^\pi e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{1}{n!}$$

*for all* $n = 0, 1, 2, \ldots$.

**Solution.** See 2001 question 2(a). ∎

**Question 2(b)** *Integrating* $e^{-z^2}$ *along a suitable rectangular contour show that*

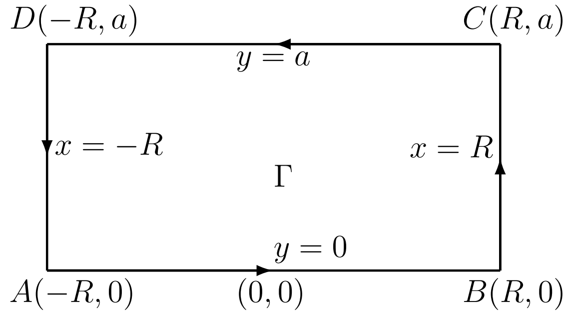
$$\int_0^\infty e^{-x^2}\cos 2bx\,dx = \frac{\sqrt{\pi}}{2}e^{-b^2}$$

**Solution.** More generally, we shall prove that

$$\int_0^\infty e^{-\lambda x^2}\cos 2a\lambda x\,dx = \frac{\sqrt{\pi}}{2}\lambda^{-\frac{1}{2}}e^{-\lambda a^2}$$

then $\lambda = 1, a = b$ will give us the desired result.

Our $f(z) = e^{-\lambda z^2}$ and the contour is $\Gamma$, the rectangle $ABCD$ where $A = (-R, 0), B = (R, 0), C = (R, a), D = (-R, a)$ oriented in the anticlockwise direction.




Since $e^{-\lambda z^2}$ is an entire function, and therefore has no poles, $\oint_\Gamma e^{-\lambda z^2}\,dz = 0$.

Now we compute the integrals along the four sides.

1.
$$\int_{BC} e^{-\lambda z^2}\,dz = \int_0^a e^{-\lambda(R+iy)^2}\,i\,dy$$
because $z = R + iy$ on $BC$ and $0 \le y \le a$. Thus
$$\left|\int_{BC} e^{-\lambda z^2}\,dz\right| \le e^{-\lambda R^2}\int_0^a e^{\lambda y^2}\,dy = \text{constant} \times e^{-\lambda R^2}$$
Thus since $e^{-\lambda R^2} \to 0$ as $R \to \infty$, $\int_{BC} e^{-\lambda z^2}\,dz \to 0$ as $R \to \infty$.

2. A similar argument shows that $\int_{DA} e^{-\lambda z^2}\,dz \to 0$ as $R \to \infty$.

3.
$$\lim_{R\to\infty}\int_{AB} e^{-\lambda z^2}\,dz = \int_{-\infty}^{\infty} e^{-\lambda x^2}\,dx$$
as $z = x$ on $AB$.

4.
$$\lim_{R\to\infty}\int_{CD} e^{-\lambda z^2}\,dz = \int_{\infty}^{-\infty} e^{-\lambda(x+ia)^2}\,dx$$
as $z = x + ia$ on $CD$ and orientation is from $C$ to $D$. Thus
$$\begin{aligned}\lim_{R\to\infty}\int_{CD} e^{-\lambda z^2}\,dz &= \int_{\infty}^{-\infty} e^{-\lambda(x^2-a^2)}e^{-2ia\lambda x}\,dx \\ &= \int_{-\infty}^{\infty} -e^{-\lambda(x^2-a^2)}\cos(2a\lambda x)\,dx + i\int_{-\infty}^{\infty} e^{-\lambda(x^2-a^2)}\sin(2a\lambda x)\,dx\end{aligned}$$

Now
$$\begin{aligned}0 &= \lim_{R\to\infty}\int_\Gamma f(z)\,dz = \lim_{R\to\infty}\int_\Gamma e^{-\lambda z^2}\,dz \\ &= \lim_{R\to\infty}\left[\int_{AB} e^{-\lambda z^2}\,dz + \int_{BC} e^{-\lambda z^2}\,dz + \int_{CD} e^{-\lambda z^2}\,dz + \int_{DA} e^{-\lambda z^2}\,dz\right] \\ &= \int_{-\infty}^{\infty} e^{-\lambda x^2}\,dx - e^{\lambda a^2}\int_{-\infty}^{\infty} e^{-\lambda x^2}\cos(2a\lambda x)\,dx + ie^{\lambda a^2}\int_{-\infty}^{\infty} e^{-\lambda x^2}\sin(2a\lambda x)\,dx\end{aligned}$$

Equating real and imaginary parts, we get

$$\begin{aligned}
\int_{-\infty}^{\infty} e^{-\lambda x^2} \sin(2a\lambda x)\, dx &= 0 \\
\int_{-\infty}^{\infty} e^{-\lambda x^2} \cos(2a\lambda x)\, dx &= e^{-\lambda a^2} \int_{-\infty}^{\infty} e^{-\lambda x^2}\, dx \\
& \quad \text{(Substituting } X = \sqrt{\lambda} x\text{)} \\
&= 2\int_0^{\infty} e^{-X^2} \lambda^{-\frac{1}{2}} dX = \sqrt{\pi}\lambda^{-\frac{1}{2}} \\
\Rightarrow \int_0^{\infty} e^{-\lambda x^2} \cos(2a\lambda x)\, dx &= \frac{1}{2}\int_{-\infty}^{\infty} e^{-\lambda x^2} \cos(2a\lambda x)\, dx \;=\; \frac{1}{2} e^{-\lambda a^2} \lambda^{-\frac{1}{2}} \sqrt{\pi}
\end{aligned}$$

This completes the proof. ∎

**Question 2(c)** *Find the function $f(z)$ analytic within the unit circle which takes the values $\dfrac{a - \cos\theta + i\sin\theta}{a^2 - 2a\cos\theta + 1}, 0 \leq \theta \leq 2\pi$ on the circle.*

**Solution.** Since $f(z)$ is analytic within $|z| < 1$, the Maclaurin series of $f(z)$ is given by

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0) z^n}{n!}, \quad \text{where } f^{(n)}(0) = \frac{n!}{2\pi i} \int_{|z|=1} \frac{f(z)\, dz}{z^{n+1}}$$

We are given that on $|z| = 1$, $f(z) = \dfrac{a - \cos\theta + i\sin\theta}{a^2 - 2a\cos\theta + 1}$ and we know that on $|z| = 1$, $z = e^{i\theta}, dz = ie^{i\theta} d\theta, 0 \leq \theta \leq 2\pi$, therefore

$$f(z) = \frac{a - \frac{1}{z}}{a^2 - a(z + \frac{1}{z}) + 1} = \frac{a - \frac{1}{z}}{(a - \frac{1}{z})(a - z)} = \frac{1}{a - z} \text{ on } |z| = 1$$

Now we use the Maclaurin series to compute the value of $f$ inside the unit circle.

$$\begin{aligned}
f^{(n)}(0) &= \frac{n!}{2\pi i} \int_{|z|=1} \frac{dz}{(a - z) z^{n+1}} \\
&= \frac{n!}{2\pi i} \int_0^{2\pi} \frac{ie^{i\theta} e^{-(n+1)i\theta}}{a - e^{i\theta}}\, d\theta \\
&= \frac{n!}{2\pi a} \int_0^{2\pi} e^{-ni\theta} \left(1 - \frac{e^{i\theta}}{a}\right)^{-1} d\theta \\
&= \frac{n!}{2\pi a} \int_0^{2\pi} e^{-ni\theta} \left(1 + \frac{e^{i\theta}}{a} + \frac{e^{2i\theta}}{a^2} + \ldots \frac{e^{in\theta}}{a^n} + \ldots\right) d\theta
\end{aligned}$$

Since $\displaystyle\int_0^{2\pi} e^{ik\theta}\, d\theta = \frac{e^{ik\theta}}{ik}\Big]_0^{2\pi} = 0$ for $k \neq 0$, and $\displaystyle\int_0^{2\pi} d\theta = 2\pi$, it follows that

$$f^{(n)}(0) = \frac{n!}{2\pi a} \cdot \frac{1}{a^n} \cdot 2\pi = \frac{n!}{a^{n+1}}$$

Consequently

$$f(z) = \sum_{n=0}^{\infty} \frac{z^n}{a^{n+1}} = \frac{1}{a} + \frac{z}{a^2} + \ldots = \frac{1}{a}\left(1 - \frac{z}{a}\right)^{-1} = \frac{1}{a-z}$$

We need $a > 1$ so that $|\frac{z}{a}| < 1$ on $|z| \leq 1$. ∎