UPSC Civil Services Main 1998 - Mathematics Complex Analysis

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Mathura

Question 1(a) *Show that the function*

$$f(z) = \begin{cases} \dfrac{x^3(1+i) - y^3(1-i)}{x^2+y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

is continuous and C-R conditions are satisfied at $z = 0$, but $f'(z)$ does not exist at $z = 0$.

Solution. Let $f(z) = u + iv$, then $u = \dfrac{x^3 - y^3}{x^2+y^2}, v = \dfrac{x^3+y^3}{x^2+y^2}$ for $z \neq 0$, and $u(0,0) = v(0,0) = 0$.

$$\begin{aligned} \frac{\partial u}{\partial x}(0,0) &= \lim_{h\to 0} \frac{u(h,0) - u(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2} - 0}{h} = 1 \\ \frac{\partial u}{\partial y}(0,0) &= \lim_{k\to 0} \frac{u(0,k) - u(0,0)}{k} = \lim_{k\to 0} \frac{\frac{-k^3}{k^2} - 0}{k} = -1 \\ \frac{\partial v}{\partial x}(0,0) &= \lim_{h\to 0} \frac{v(h,0) - v(0,0)}{h} = \lim_{h\to 0} \frac{\frac{h^3}{h^2} - 0}{h} = 1 \\ \frac{\partial v}{\partial y}(0,0) &= \lim_{k\to 0} \frac{v(0,k) - v(0,0)}{k} = \lim_{k\to 0} \frac{\frac{k^3}{k^2} - 0}{k} = 1 \end{aligned}$$

Thus $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$ at $(0,0)$, i.e. the Cauchy Riemann equations are satisfied at $(0,0)$.

$f(z)$ is clearly continuous at $z = 0$, because

$$\begin{aligned} |u(x,y) - u(0,0)| &= \left|\frac{x^3 - y^3}{x^2+y^2}\right| = \left|\frac{r^3(\cos^3\theta - \sin^3\theta)}{r^2}\right| \leq 2\sqrt{x^2+y^2} \\ |v(x,y) - v(0,0)| &\leq 2\sqrt{x^2+y^2} \end{aligned}$$

Thus u, v are continuous at $(0,0)$, so $f(z)$ is continuous at $(0,0)$.

If $f(z)$ is to be differentiable at 0, then

$$\lim_{z\to 0}\frac{f(z)-0}{z} = \lim_{x\to 0, y\to 0}\frac{(x^3-y^3)+i(x^3+y^3)}{(x^2+y^2)(x+iy)} = \lim_{x\to 0, y\to 0}\frac{(x^3+iy^3)(1+i)(x-iy)}{(x^2+y^2)^2}$$

should exist and it should be equal to $\dfrac{\partial u}{\partial x}(0,0) + i\dfrac{\partial v}{\partial x}(0,0) = 1+i$.

But if we take the limit along $y = x$, then

$$\lim_{z\to 0}\frac{f(z)-0}{z} = \lim_{x\to 0}\frac{(x^3+ix^3)(1+i)(x-ix)}{(2x^2)^2} = \frac{1+i}{2}$$

Therefore $f(z)$ is not differentiable at $z = 0$. ∎

Question 1(b) *Find the Laurent expansion of* $\dfrac{z}{(z+1)(z+2)}$ *about the singularity* $z = -2$. *Specify the region of convergence and nature of singularity at* $z = -2$.

Solution. Clearly

$$\begin{aligned} f(z) &= \frac{z}{(z+1)(z+2)} = \frac{2}{z+2} - \frac{1}{z+1} = \frac{2}{z+2} + \frac{1}{1-(z+2)} \\ &= \frac{2}{z+2} + \sum_{n=0}^{\infty}(z+2)^n \text{ for } |z+2| < 1 \qquad (*) \end{aligned}$$

The function satisfies the requirements of Laurent's theorem in the region $0 < |z+2| < 1$ and the right hand side of $(*)$ represents the Laurent series of $f(z)$, which converges for $|z+2| < 1$, because we have a singularity at $z = -1$ which lies on $|z+2| = 1$. The Laurent series expansion $(*)$ shows that $f(z)$ has a simple pole at $z = -2$, where its residue is 2. ∎

Question 1(c) *By using the integral representation of* $f^{(n)}(0)$*, prove that*

$$\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i}\oint_C \frac{x^n e^{xz}}{n! z^{n+1}}\, dz$$

Hence show that

$$\sum_{n=0}^{\infty}\left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi}\int_0^{2\pi} e^{2x\cos\theta}\, d\theta$$

Solution. It is easily deducible from Cauchy's Integral formula that if $f(z)$ is analytic within and on a simple closed contour C and z_0 is a point in the interior of C, then

$$f^{(n)}(z_0) = \frac{n!}{2\pi i}\oint_C \frac{f(z)}{(z-z_0)^{n+1}},dz$$

Let $f(z) = e^{xz}$ (Here x is not $\operatorname{Re} x$ but a parameter), then $f(z)$ is an entire function and therefore

$$f^{(n)}(0) = x^n = \frac{n!}{2\pi i} \oint_C \frac{e^{xz}}{z^{n+1}}, dz$$

where C is any closed contour containing 0 in its interior. Hence

$$\left(\frac{x^n}{n!}\right)^2 = \frac{x^n}{(n!)^2} \frac{n!}{2\pi i} \oint_C \frac{e^{xz}}{z^{n+1}}, dz = \frac{1}{2\pi i} \oint_C \frac{x^n e^{xz}}{n! z^{n+1}}\, dz$$

as required.

We take C to be the unit circle for convenience. Then

$$\sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i} \sum_{n=0}^{\infty} \oint_C \frac{x^n e^{xz}}{n! z^{n+1}}\, dz = \frac{1}{2\pi i} \oint_C \sum_{n=0}^{\infty} \frac{x^n e^{xz}}{n! z^{n+1}}\, dz$$

Interchange of summation and integral is justified. Thus

$$\sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i} \int_{|z|=1} \frac{e^{xz}}{z} \sum_{n=0}^{\infty} \frac{(\frac{x}{z})^n}{n!}\, dz = \frac{1}{2\pi i} \int_{|z|=1} \frac{e^{xz}}{z} e^{\frac{x}{z}}\, dz$$

Put $z = e^{i\theta}$ so that $dz = ie^{i\theta}\, d\theta$ and

$$\sum_{n=0}^{\infty} \left(\frac{x^n}{n!}\right)^2 = \frac{1}{2\pi i} \int_0^{2\pi} \frac{e^{x(e^{i\theta}+e^{-i\theta})}}{e^{i\theta}} ie^{i\theta}\, d\theta = \frac{1}{2\pi} \int_0^{2\pi} e^{2x\cos\theta}\, d\theta$$

as required. ■

Question 2(a) *Prove that all roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.*

Solution. See 2006 question 2(b). ■

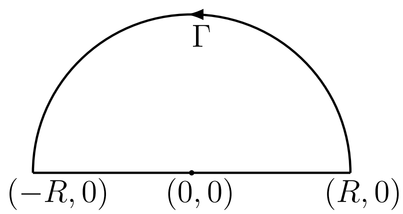
Question 2(b) *By integrating around a suitable contour show that*

$$\int_0^{\infty} \frac{x \sin mx}{x^4 + a^4}\, dx = \frac{\pi}{4b^2} e^{-mb} \sin mb$$

where $b = \frac{a}{\sqrt{2}}$.

Solution.

Let $f(z) = \dfrac{ze^{imz}}{z^4+a^4}$. We consider the integral $\int_\gamma f(z)\,dz$ where γ is the contour consisting of the line joining $(-R,0)$ and $(R,0)$ and Γ, the arc of the circle of radius R and center $(0,0)$ lying in the upper half plane.



$$\left|\int_\Gamma f(z)\,dz\right| = \left|\int_0^\pi \frac{Re^{i\theta}e^{imR(\cos\theta+i\sin\theta)}}{z^4+a^4}Rie^{i\theta}\,d\theta\right| \le \frac{R^2}{R^4-a^4}\pi$$

because $|z^4+a^4| \ge |z|^4 - |a^4| = R^4 - a^4$ on Γ, and $e^{-mR\sin\theta} \le 1$ as $\sin\theta > 0$ for $0 < \theta < \pi$.

Thus $\displaystyle\int_\Gamma f(z)\,dz \to 0$ as $R \to \infty$ and

$$\lim_{R\to\infty}\int_\gamma f(z)\,dz = \int_{-\infty}^{\infty}\frac{xe^{imx}}{x^4+a^4}\,dx$$

But by Cauchy's residue theorem $\displaystyle\int_\gamma f(z)\,dz = 2\pi i \times$ (the sum of the residues of poles of $f(z)$ inside γ). The poles of $f(z)$ are simple poles at $\pm ae^{\frac{\pi i}{4}}, \pm ae^{\frac{3\pi i}{4}}$, out of which $ae^{\frac{\pi i}{4}}, ae^{\frac{3\pi i}{4}}$ are inside γ.

Residue at $z = ae^{\frac{\pi i}{4}}$ is $\dfrac{ae^{\frac{\pi i}{4}}e^{ima\,e^{\frac{\pi i}{4}}}}{4a^3e^{\frac{3\pi i}{4}}}$. Residue at $z = ae^{\frac{3\pi i}{4}}$ is $\dfrac{ae^{\frac{3\pi i}{4}}e^{ima\,e^{\frac{3\pi i}{4}}}}{4a^3e^{\frac{9\pi i}{4}}}$.

$$\begin{aligned}
\text{Sum of residues} &= \frac{i}{4a^2}\left[-e^{ima(cos\frac{\pi}{4}+i\sin\frac{\pi}{4})} + e^{ima(cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4})}\right] \\
&= \frac{i}{4a^2}\left[-e^{\frac{ma}{\sqrt{2}}(i-1)} + e^{\frac{ma}{\sqrt{2}}(-i-1)}\right] \\
&= \frac{ie^{-\frac{ma}{\sqrt{2}}}}{4a^2}\left(-2i\sin\frac{ma}{\sqrt{2}}\right) = \frac{e^{-\frac{ma}{\sqrt{2}}}}{2a^2}\sin\frac{ma}{\sqrt{2}}
\end{aligned}$$

Thus

$$\int_{-\infty}^{\infty}\frac{xe^{imx}}{x^4+a^4}\,dx = 2\pi i\frac{e^{-\frac{ma}{\sqrt{2}}}}{2a^2}\sin\frac{ma}{\sqrt{2}}$$

Taking imaginary parts of both sides, we get

$$\int_{-\infty}^{\infty}\frac{x\sin mx}{x^4+a^4}\,dx = 2\int_0^\infty\frac{x\sin mx}{x^4+a^4}\,dx = \frac{\pi e^{-\frac{ma}{\sqrt{2}}}}{a^2}\sin\frac{ma}{\sqrt{2}} = \frac{\pi e^{-mb}}{2b^2}\sin mb$$

where $b = \frac{a}{\sqrt{2}}$. Thus

$$\int_0^\infty\frac{x\sin mx}{x^4+a^4}\,dx = \frac{\pi e^{-mb}}{4b^2}\sin mb$$

as required. ∎

Question 2(c) *Using the residue theorem evaluate* $\displaystyle\int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta}$.

Solution. We put $z = e^{i\theta}$, so that $d\theta = \frac{dz}{iz}, \cos\theta = \frac{1}{2}(z + \frac{1}{z}), \sin\theta = \frac{1}{2i}(z - \frac{1}{z})$. Thus

$$\begin{aligned} I &= \int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} \\ &= \oint_{|z|=1} \frac{dz}{iz[3 - (z + \frac{1}{z}) + \frac{1}{2i}(z - \frac{1}{z})]} \\ &= 2\oint_{|z|=1} \frac{dz}{6iz - 2iz^2 - 2i + z^2 - 1} \\ &= 2\oint_{|z|=1} \frac{dz}{(1-2i)(z + \frac{i}{1-2i})(z + \frac{5i}{1-2i})} \end{aligned}$$

Clearly $(6iz - 2iz^2 - 2i + z^2 - 1)^{-1}$ has two simple poles $-\frac{i}{1-2i}$ and $-\frac{5i}{1-2i}$ of which only $-\frac{i}{1-2i}$ lies inside $|z| = 1$. The residue at this pole is $\displaystyle\lim_{z \to -\frac{i}{1-2i}} \frac{z + \frac{i}{1-2i}}{(1-2i)(z + \frac{i}{1-2i})(z + \frac{5i}{1-2i})} = \frac{1}{4i}$. Thus by Cauchy's residue theorem

$$I = \int_0^{2\pi} \frac{d\theta}{3 - 2\cos\theta + \sin\theta} = 2 \cdot 2\pi i \cdot \frac{1}{4i} = \pi$$

∎