UPSC Civil Services Main 1999 - Mathematics Complex Analysis

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Question 1(a) *Examine the nature of the function*

$$f(z) = \frac{x^2y^5(x+iy)}{x^4+y^{10}}, z \neq 0, f(0) = 0$$

in a region including the origin and hence show that the Cauchy-Riemann equations are satisfied at the origin, but $f(z)$ is not analytic there.

Solution.

$$\begin{aligned} u(x,y) &= \operatorname{Re} f(z) = \begin{cases} \frac{x^3y^5}{x^4+y^{10}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \\ v(x,y) &= \operatorname{Im} f(z) = \begin{cases} \frac{x^2y^6}{x^4+y^{10}}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases} \end{aligned}$$

Now $\dfrac{u(x,0)-u(0,0)}{x} = 0 = \dfrac{v(0,y)-v(0,0)}{y}$, therefore $u_x(0,0) = v_y(0,0) = 0$. Similarly $u_y(0,0) = 0 = -v_x(0,0)$. Thus the Cauchy-Riemann equations are satisfied at $(0,0)$.

However $f(z)$ is not analytic at $(0,0)$ because $\displaystyle\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \lim_{z\to 0} \frac{x^2y^5}{x^4+y^{10}}$ does not exist — when we take $y^5 = mx^2$, then $\displaystyle\lim_{z\to 0} \frac{f(z)-f(0)}{z} = \frac{m}{1+m^2}$ which is different for different values of m. ∎

Additional notes: Let $z \neq 0$. It can be calculated that

$$\frac{\partial u}{\partial x} = \frac{3x^2y^{15} - x^6y^5}{(x^4+y^{10})^2} \qquad \frac{\partial v}{\partial y} = \frac{-4x^2y^{15} + 6x^6y^5}{(x^4+y^{10})^2}$$
$$\frac{\partial v}{\partial x} = \frac{2xy^{16} - 2x^5y^6}{(x^4+y^{10})^2} \qquad \frac{\partial u}{\partial y} = \frac{5x^7y^4 - 5x^3y^{14}}{(x^4+y^{10})^2}$$

Now $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Leftrightarrow 3x^2y^{15} - x^6y^5 = -4x^2y^{15} + 6x^6y^5 \Leftrightarrow x^2y^{15} = x^6y^5 \Leftrightarrow x^4 = y^{10}$ or $x = 0$ or $y = 0$.

Also, $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ when $x^4 = y^{10}$ or $x = 0$ or $y = 0$. Thus the Cauchy-Riemann equations are satisfied at all those z for which $x^4 = y^{10}$ or $x = 0$ or $y = 0$. But $f(z)$ is not analytic at any of these points because $f(z)$ is not differentiable in any neighborhood of these points, as we can find points in every neighborhood which are not of this kind, so there are no neighborhoods in which the Cauchy Riemann equations are satisfied everywhere.

Question 1(b) *For the function* $f(z) = \dfrac{-1}{z^2 - 3z + 2}$, *find the Laurent series for the domain (i)* $1 < |z| < 2$ *(ii)* $|z| > 2$.

Show further that $\oint_C f(z)\,dz = 0$ *where* C *is any closed contour enclosing the points* $z = 1$ *and* $z = 2$.

Solution. $f(z) = \dfrac{1}{z-1} - \dfrac{1}{z-2}$

(i) $1 < |z| < 2 \Rightarrow |\frac{1}{z}| < 1, |\frac{z}{2}| < 1$.

$$\begin{aligned} f(z) &= \frac{1}{z}\Big(1 - \frac{1}{z}\Big)^{-1} + \frac{1}{2}\Big(1 - \frac{z}{2}\Big)^{-1} \\ &= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} + \frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n} \\ &= \sum_{n=1}^{\infty}\frac{1}{z^n} + \sum_{n=0}^{\infty}\frac{z^n}{2^{n+1}} \end{aligned}$$

(ii) $|z| > 2 \Rightarrow |\frac{1}{z}| < 1, |\frac{2}{z}| < 1$

$$\begin{aligned} f(z) &= \frac{1}{z}\Big(1 - \frac{1}{z}\Big)^{-1} - \frac{1}{z}\Big(1 - \frac{2}{z}\Big)^{-1} \\ &= \frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n} - \frac{1}{z}\sum_{n=0}^{\infty}\frac{2^n}{z^n} \\ &= \sum_{n=0}^{\infty}\frac{1-2^n}{z^{n+1}} \end{aligned}$$

$$
\begin{aligned}
\oint_C f(z)\,dz &= \oint_C \Big(\frac{1}{z-1} - \frac{1}{z-2}\Big)dz \\
&= 2\pi i\big[\text{residue of } \tfrac{1}{z-1} \text{ at } z = 1 - \text{residue of } \tfrac{1}{z-2} \text{ at } z = 2\big] \\
&= 2\pi i[1-1] = 0
\end{aligned}
$$

∎

Question 1(c) *Show that the transformation* $w = \dfrac{2z+3}{z-4}$ *transforms the circle* $x^2+y^2-4x = 0$ *into the straight line* $4u+3=0$ *where* $w = u+iv$.

Solution. The point $z = 4$ goes to the point at ∞, showing that the given circle $0 = x^2+y^2-4x = z\overline{z} - 4(\frac{z+\overline{z}}{2}) = z\overline{z} - 2z - 2\overline{z} = 0$ is mapped onto a line, as $z = 4$ lies on it.

Now $zw - 4w = 2z+3 \Rightarrow zw - 2z = 3+4w \Rightarrow z = \dfrac{3+4w}{w-2}$. Thus the circle $z\overline{z} - 2z - 2\overline{z} = 0$ goes to

$$
\begin{aligned}
0 &= \frac{3+4w}{w-2}\,\frac{3+4\overline{w}}{\overline{w}-2} - 2\frac{3+4w}{w-2} - 2\frac{3+4\overline{w}}{\overline{w}-2} = 0 \\
\Rightarrow 0 &= 9 + 12w + 12\overline{w} + 16w\overline{w} - 2(3+4w)(\overline{w}-2) - 2(3+4\overline{w})(w-2) \\
\Rightarrow 0 &= 9 + 12w + 12\overline{w} + 16w\overline{w} - 6\overline{w} + 12 + 16w - 8w\overline{w} - 6w + 12 + 16\overline{w} - 8w\overline{w} \\
&= 33 + 22w + 22\overline{w} \\
0 &= 2(w+\overline{w}) + 3
\end{aligned}
$$

Thus $4u+3=0$, as required. ∎

Alternate solution: The given circle is $|z-2| = 2 \Rightarrow z = 2 + 2e^{i\theta}$. Substituting in transformation expression,

$$
\begin{aligned}
w &= \frac{2z+3}{z-4} = \frac{4+4e^{i\theta}+3}{2+2e^{i\theta}-4} = \frac{7+4e^{i\theta}}{2(e^{i\theta}-1)} = \frac{(7+4e^{i\theta})(e^{-i\theta}-1)}{2(e^{i\theta}-1)(e^{-i\theta}-1)} \\
&= \frac{7e^{-i\theta} - 4e^{i\theta} - 3}{2(2 - e^{i\theta} - e^{-i\theta})} = \frac{7(\cos\theta - i\sin\theta) - 4(\cos\theta + i\sin\theta) - 3}{2(2-2\cos\theta)} \\
&= \frac{3\cos\theta - 3 - 11i\sin\theta}{4(1-\cos\theta)} = -\frac{3}{4} - i\frac{11\sin\theta}{4(1-\cos\theta)}
\end{aligned}
$$

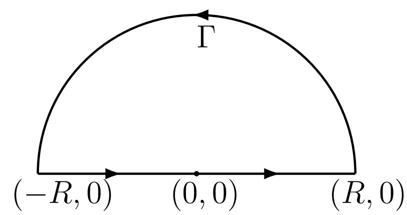
Thus $u = -\frac{3}{4} \Rightarrow 4u+3=0$, hence all points on the circle $|z-2| = 2$ are mapped onto the line $4u+3=0$.

Question 2(a) *Using the Residue Theorem show that*

$$
\int_{-\infty}^{\infty} \frac{x\sin ax}{x^4+4}\,dx = \frac{\pi}{2}e^{-a}\sin a \quad (a > 0)
$$

Solution.

We consider $I = \int_\gamma f(z)\,dz$ where $f(z) = \dfrac{ze^{iaz}}{z^4+4}$ and the contour γ consists of Γ a semicircle of radius R with center $(0,0)$ lying in the upper half plane bounded by the real axis as shown.



Thus by Cauchy's residue theorem, $\int_\gamma f(z)\,dz = 2\pi i$(sum of residues at poles of $f(z)$ inside γ).

Clearly $f(z)$ has simple poles at $z^4 = 4e^{(2n+1)\pi i}$ for $n = 0, 1, 2, 3$, or $z = \sqrt{2}e^{\frac{\pi i}{4}}, \sqrt{2}e^{\frac{3\pi i}{4}}, \sqrt{2}e^{\frac{5\pi i}{4}}, \sqrt{2}e^{\frac{7\pi i}{4}}$. Out of these only the poles $\sqrt{2}e^{\frac{\pi i}{4}}, \sqrt{2}e^{\frac{3\pi i}{4}}$ lie inside γ.

Residue at $\sqrt{2}e^{\frac{\pi i}{4}}$ is $\left(\dfrac{ze^{iaz}}{\frac{d}{dz}(z^4+4)}\right)$ at $z = \sqrt{2}e^{\frac{\pi i}{4}}$, which is $\dfrac{\alpha e^{ia\alpha}}{4\alpha^3}$ where $\alpha = \sqrt{2}e^{\frac{\pi i}{4}} = 1+i$.

Residue at $\sqrt{2}e^{\frac{3\pi i}{4}}$ is $\dfrac{e^{ia\beta}}{4\beta^2}$ where $\beta = \sqrt{2}e^{\frac{3\pi i}{4}} = -1+i$.

Sum of these residues is

$$\begin{aligned}\frac{1}{4}\Big[\frac{e^{ia\alpha}}{\alpha^2} + \frac{e^{ia\beta}}{\beta^2}\Big] &= \frac{1}{4}\Big[\frac{e^{ia(1+i)}}{2i} + \frac{e^{ia(-1+i)}}{(-2i)}\Big] \\ &= \frac{e^{-a}}{8i}\big[e^{ia} - e^{-ia}\big] = \frac{e^{-a}\sin a}{4}\end{aligned}$$

Thus $\displaystyle\int_\gamma \frac{ze^{iaz}\,dz}{z^4+4} = 2\pi i\frac{e^{-a}\sin a}{4}$. Now

$$\left|\int_\Gamma \frac{ze^{iaz}\,dz}{z^4+4}\right| = \left|\int_0^\pi \frac{Re^{i\theta}e^{iaRe^{i\theta}}}{z^4+4} iRe^{i\theta}\,d\theta\right| \leq \frac{R^2}{R^4-4}\int_0^\pi e^{-aR\sin\theta}\,d\theta \leq \frac{\pi R^2}{R^4-4}$$

because $|z^4+4| \geq |z^4| - 4 = R^4 - 4$ on Γ, and $e^{-aR\sin\theta} \leq 1$ as $\sin\theta \geq 0$ on $[0,\pi]$. Thus $\int_\Gamma f(z)\,dz \to 0$ as $R \to \infty$. Thus

$$\int_{-\infty}^{\infty} \frac{xe^{iax}}{x^4+4}\,dx = \lim_{R\to\infty}\int_\gamma \frac{ze^{iaz}}{z^4+4}\,dz = \frac{e^{-a}\sin a}{4}2\pi i$$

Taking the imaginary parts of both sides, we get

$$\int_{-\infty}^{\infty} \frac{x\sin ax}{x^4+4}\,dx = \frac{\pi e^{-a}\sin a}{2}$$

as required. ∎

Question 2(b) *The function $f(z)$ has a double pole at $z = 0$ with residue 2, a simple pole at $z = 1$ with residue 2, is analytic at all other finite points of the plane and is bounded as $|z| \to \infty$. If $f(2) = 5$ and $f(-1) = 2$, find $f(z)$.*

Solution. Since $f(z)$ has only poles as singularities in the extended complex plane, it is well known that $f(z)$ has to be a rational function. Since $f(z)$ has a double pole at $z = 0$ and a simple pole at $z = 1$, it has to be of the form $f(z) = \dfrac{\phi(z)}{z^2(z-1)}$. where $\phi(z)$ is a polynomial such that $\phi(0) \neq 0, \phi(1) \neq 0$. Moreover degree of $\phi(z)$ is ≤ 3 as we are given that $f(z)$ is bounded as $z \to \infty$. Let $\phi(z) = a_0 + a_1 z + a_2 z^2 + a_3 z^3$. Then

$$f(2) = 5 \Rightarrow \frac{a_0 + 2a_1 + 4a_2 + 8a_3}{4} = 5 \qquad (1)$$

$$f(-1) = 2 \Rightarrow \frac{a_0 - a_1 + a_2 - a_3}{-2} = 2 \qquad (2)$$

Residue of $f(z)$ at $z = 1$ is $\lim_{z \to 1} \dfrac{(z-1)\phi(z)}{z^2(z-1)} = \phi(1)$. This value is given to be 2, so

$$a_0 + a_1 + a_2 + a_3 = 2 \qquad (3)$$

Residue of $f(z)$ at $z = 0$ is given by $\dfrac{1}{1!}\dfrac{d}{dz}\left(\dfrac{\phi(z)}{z-1}\right)$ at $z = 0$, or $\dfrac{(z-1)(\phi'(z)) - \phi(z)}{(z-1)^2} = -a_1 - a_0$. Since this is given to be 2,

$$-a_0 - a_1 = 2 \qquad (4)$$

Adding $(2), (3)$ we get $2a_0 + 2a_2 = -2 \Rightarrow a_2 = -1 - a_0$. Substituting $a_2 = -1 - a_0, a_1 = -a_0 - 2$ in (1), we get $a_0 - 2a_0 - 4 - 4 - 4a_0 + 8a_3 = 20 \Rightarrow 8a_3 = 5a_0 + 28$. Substituting in (3), we have $a_0 - a_0 - 2 - 1 - a_0 + \frac{5a_0+28}{8} = 2 \Rightarrow -3a_0 + 28 = 40 \Rightarrow a_0 = -4 \Rightarrow a_1 = 2, a_2 = 3, a_3 = 1$.

Hence $f(z) = \dfrac{-4 + 2z + 3z^2 + z^3}{z^2(z-1)}$ is the desired function.

Note: If $f(z)$ has only poles in $\mathbb{C} \cup \infty$, then it is a rational function. If $\phi_1(z), \phi_2(z), \dots, \phi_r(z)$ are principal parts of $f(z)$ at the poles$z_1, z_2, \dots, z_r$ and $\psi(z)$ is the principal part of $f(z)$ at ∞, then $f(z) - \sum_{j=1}^r \phi_j(z) - \psi(z)$ being bounded and analytic in $\mathbb{C} \cup \infty$ is constant $\Rightarrow f(z) = \sum_{j=1}^r \phi_j(z) + \psi(z) + C$. Thus $f(z)$ is a rational function, as each $\phi_j(z)$ is a rational function and $\psi(z)$ is a polynomial. ∎

Question 2(c) *What kind of singularities do the following functions have?*

1. $\dfrac{1}{1 - e^z}$ *at* $z = 2\pi i$.
2. $\dfrac{1}{\sin z - \cos z}$ *at* $z = \frac{\pi}{4}$.

3. $\dfrac{\cot \pi z}{(z-a)^2}$ *at* $z = a$ *and* $z = \infty$. *What happens when* a *is an integer (including* $a = 0$*)?*

Solution.

1. Clearly $e^z - 1 = e^{z-2\pi i} - 1 = (z - 2\pi i) + \dfrac{(z-2\pi i)^2}{2!} + \dfrac{(z-2\pi i)^3}{3!} + \ldots$, showing that $e^z - 1$ has a simple zero at $z = 2\pi i$. Thus the given function $\frac{1}{1-e^z}$ has a simple pole at $z = 2\pi i$. Now residue at $z = 2\pi i$ is given by

$$\lim_{z \to 2\pi i} \frac{z - 2\pi i}{1 - e^z} = -1$$

2. $f(z) = \dfrac{1}{\sin z - \cos z}$. We know that

$$\begin{aligned}
\sin z &= \frac{1}{\sqrt{2}} + \Big(z - \frac{\pi}{4}\Big)\frac{1}{\sqrt{2}} - \frac{\big(z - \frac{\pi}{4}\big)^2}{2!}\frac{1}{\sqrt{2}} + \ldots + \text{Higher powers of } \Big(z - \frac{\pi}{4}\Big) \\
\cos z &= \frac{1}{\sqrt{2}} - \Big(z - \frac{\pi}{4}\Big)\frac{1}{\sqrt{2}} - \frac{\big(z - \frac{\pi}{4}\big)^2}{2!}\frac{1}{\sqrt{2}} + \ldots + \text{Higher powers of } \Big(z - \frac{\pi}{4}\Big) \\
\Rightarrow \sin z - \cos z &= \sqrt{2}\Big(z - \frac{\pi}{4}\Big) - \sqrt{2}\frac{\big(z - \frac{\pi}{4}\big)^3}{3!} + \ldots + \text{Higher powers of } \Big(z - \frac{\pi}{4}\Big)
\end{aligned}$$

Since $\sin z - \cos z$ has a simple zero at $z = \dfrac{\pi}{4}$, the given function $\dfrac{1}{\sin z - \cos z}$ has a simple pole at $z = \dfrac{\pi}{4}$.

Residue at $z = \dfrac{\pi}{4}$ is given by $\displaystyle\lim_{z \to \frac{\pi}{4}} \frac{z - \frac{\pi}{4}}{\sin z - \cos z} = \frac{1}{\sqrt{2}}$.

3. $f(z) = \dfrac{\cot \pi z}{(z-a)^2}$. $f(z)$ has a simple pole at each $z = n, n \in \mathbb{Z}, n \neq a$, with residue $\frac{1}{(n-a)^2}$. $f(z)$ also has a pole at $z = a$, whose nature is as follows:

   (a) a is not an integer and $a \neq n + \frac{1}{2}$.
   In this case, $\cos \pi a \neq 0, \sin \pi a \neq 0$ and therefore $f(z)$ has a double pole at $z = a$. (The residue at $z = a$ is $\dfrac{d}{dz}[(z-a)^2 f(z)]_{z=a} = -\pi \csc^2 \pi a$.)

   (b) a is not an integer and $a = n + \frac{1}{2}$.
   In this case $\cos \pi z$ has a simple zero at a, and $\sin \pi z = \pm 1$, therefore $f(z)$ has a simple pole at $z = a$. (The residue at $z = a$ is $\displaystyle\lim_{z \to a} \frac{\cos \pi z}{z - a}\frac{1}{\sin \pi a} = \frac{-\pi \sin \pi a}{\sin \pi a} = -\pi$.)

(c) a is an integer.

$\sin \pi z$ has a simple zero at $z = a$ and $\cos \pi a \neq 0$, then $f(z)$ has a **triple pole** at $z = a$. The residue in this case is $-\frac{\pi}{3}$, because

$$\begin{aligned}
\sin \pi z &= (-1)^a \left[\pi(z-a) - \pi^3 \frac{(z-a)^3}{3!} + \text{Higher powers of}(z-a)\right] \\
\cos \pi z &= (-1)^a \left[1 - \pi^2 \frac{(z-a)^2}{2!} + \text{Higher powers of}(z-a)\right] \\
f(z) &= \frac{1}{(z-a)^2} \; \frac{1 - \pi^2 \frac{(z-a)^2}{2!} + \text{Higher powers of}(z-a)}{\pi(z-a)[1 - \pi^2 \frac{(z-a)^2}{3!} + \text{Higher powers of}(z-a)]} \\
&= \frac{1}{\pi(z-a)^3} \left[1 - \pi^2 \frac{(z-a)^2}{2!} + \dots\right] \left[1 + \pi^2 \frac{(z-a)^2}{3!} + \dots\right]
\end{aligned}$$

The coefficient of $\frac{1}{z-a}$ in the Laurent series of $f(z)$ (formed by multiplying the above series) is $\frac{1}{\pi}\left[-\frac{\pi^2}{2} + \frac{\pi^2}{6}\right] = -\frac{\pi}{3}$, which is the required residue.

(Note that the computation of residues was not required for this problem.)

Finally, $f(z)$ has an essential singularity at ∞, because $f(z)$ has zeros at $z = n + \frac{1}{2}, a \neq n + \frac{1}{2}$ whose limit point is ∞.

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