UPSC Civil Services Main 2000 - Mathematics Complex Analysis

Brij Bhooshan

Asst. Professor

B.S.A. College of Engg & Technology

Mathura

Question 1(a) *Show that any four given points of the complex plane can be carried by a bilinear transformation to positions* $1, -1, k, -k$ *where the value of* k *depends on the given points.*

Solution. It is known that a bilinear transformation mapping z_1, z_2, z_3 to w_1, w_2, w_3 is given by the crossratio(z_1, z_2, z_3, z) = crossratio(w_1, w_2, w_3, w), i.e.

$$\frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)} = \frac{(w - w_1)(w_2 - w_3)}{(w_1 - w_2)(w_3 - w)}$$

Now $w_1 = 1, w_2 = -1, w_3 = k$, so

$$\frac{(z - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z)} = \frac{(w - 1)(-1 - k)}{(2)(k - w)}$$

It will map z_4 to $-k$ provided k is given by

$$\frac{(z_4 - z_1)(z_2 - z_3)}{(z_1 - z_2)(z_3 - z_4)} = \frac{(-k - 1)(-1 - k)}{(2)(k - (-k)} = \frac{(k + 1)^2}{4k}$$

Clearly k depends on the points z_1, z_2, z_3, z_4. ∎

Question 2(a) *Suppose* $f(\zeta)$ *is continuous on a circle* C*. Show that* $\int_C \frac{f(s)\,ds}{s - z}$ *as* z *varies inside* C *is differentiable under the integral sign. Find the derivative. Hence or otherwise derive as integral representation for* $f'(z)$ *if* $f(z)$ *is analytic on and inside of* C*.*

Solution. If $f(z)$ is analytic on and inside C, then by Cauchy's integral formula

$$f(z) = \frac{1}{2\pi i} \int_C \frac{f(s)\, ds}{s-z}$$

Let h be a complex number so chosen that $z+h$ also lies in the interior of C. Then

$$\begin{aligned}
\frac{f(z+h)-f(z)}{h} &= \frac{1}{2\pi i}\Big[\frac{1}{h}\int_C \Big[\frac{f(s)}{s-z-h} - \frac{f(s)}{s-z}\Big] ds\Big] \\
&= \frac{1}{2\pi i}\Big[\frac{1}{h}\int_C \frac{hf(s)}{(s-z)(s-z-h)}\, ds\Big] \\
\Longrightarrow \frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\, ds &= \frac{1}{2\pi i}\int_C \Big[\frac{f(s)}{(s-z-h)(s-z)} - \frac{f(s)}{(s-z)^2}\Big] ds \\
&= \frac{1}{2\pi i}\int_C \frac{hf(s)}{(s-z-h)(s-z)^2} ds
\end{aligned}$$

Let $M = \sup_{z\in C} |f(z)|$, $l =$ arc length of C, $d = \min_{s\in C} |z-s| > 0$. Since we are interested in $h \to 0$, we can assume that $0 < |h| < d$. Now $|s-z| \geq d$, $|s-z-h| \geq |s-z|-|h| \geq d-|h|$, and therefore

$$\left|\frac{f(z+h)-f(z)}{h} - \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\, ds\right| \leq \frac{M}{2\pi}\frac{|h|}{d^2(d-|h|)} \cdot l$$

Since the right hand side of the above inequality $\to 0$ as $h \to 0$, it follows that

$$\lim_{h\to 0} \frac{f(z+h)-f(z)}{h} = \frac{1}{2\pi i}\int_C \frac{f(s)}{(s-z)^2}\, ds$$

i.e.

$$\frac{d(f(z))}{dz} = f'(z) = \frac{1}{2\pi i}\int_C \frac{d}{dz}\Big(\frac{f(s)}{s-z}\Big)\, ds$$

or

$$\frac{d}{dz}\frac{1}{2\pi i}\int_C \frac{f(s)\, ds}{s-z} = \frac{1}{2\pi i}\int_C \frac{d}{dz}\Big(\frac{f(s)}{s-z}\Big)\, ds$$

i.e. differentiation under the integral sign is valid. The representation for $f'(z)$ is given by

$$f'(z) = \frac{1}{2\pi i}\int_C \frac{f(s)\, ds}{(s-z)^2}$$

∎