UPSC Civil Services Main 2001 - Mathematics Complex Analysis

Brij Bhooshan

Asst. Professor

B.S.A. College of Engg & Technology

Mathura

Question 1(a) *Prove that the Riemann zeta function ζ defined by $\zeta(z) = \sum_{n=1}^{\infty} n^{-z}$ converges for $\operatorname{Re} z > 1$ and converges uniformly for $\operatorname{Re} z > 1 + \epsilon$ where ϵ is arbitrarily small.*

Solution.

$$\left|\frac{1}{n^z}\right| = \left|\frac{1}{n^x \cdot n^{iy}}\right| = \left|\frac{1}{n^x}\right| \qquad \because \left|\frac{1}{n^{iy}}\right| = \left|\frac{1}{e^{iy \log n}}\right| = 1$$

Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^x}$ converges for $x > 1$, it follows that $\sum_{n=1}^{\infty} n^{-z}$ converges absolutely for $\operatorname{Re} z > 1$.

If $\operatorname{Re} z \geq 1 + \epsilon$, then $\frac{1}{n^x} \leq \frac{1}{n^{1+\epsilon}}$ and

$$\sum_{n=1}^{\infty} |n^{-z}| \leq \sum_{n=1}^{\infty} \frac{1}{n^{1+\epsilon}}$$

for $\operatorname{Re} z \geq 1 + \epsilon$. Weierstrass' M-test gives that the given series converges uniformly and absolutely for $\operatorname{Re} z \geq 1 + \epsilon$. ∎

Question 2(a) *Find the Laurent series for the function $e^{\frac{1}{z}}$ in $0 < |z| < \infty$. Using the expansion show that*

$$\frac{1}{\pi} \int_0^{\pi} e^{\cos\theta} \cos(\sin\theta - n\theta)\, d\theta = \frac{1}{n!}$$

$n = 1, 2, \ldots$.

Solution. Clearly $e^{\frac{1}{z}}$ is analytic in $0 < |z| < \infty$ and satisfies requirements of Laurent's expansion, and we have

$$e^{\frac{1}{z}} = \sum_{n=-\infty}^{\infty} a_n z^n, \text{ where } a_n = \frac{1}{2\pi i} \int_{|z|=1} \frac{e^{\frac{1}{z}}}{z^{n+1}}\, dz \qquad (*)$$

Note — $z = 0$ is an essential singularity, therefore we have infinitely many terms with negative exponents. In the expression for a_n we could have taken any disc, we have taken $|z| = 1$ for convenience.

Put $z = e^{i\theta}$ in $(*)$, $dz = ie^{i\theta}\,d\theta$, we get

$$\begin{aligned} a_n &= \frac{1}{2\pi i}\int_0^{2\pi} \frac{e^{\cos\theta - i\sin\theta}}{e^{i(n+1)\theta}}\, ie^{i\theta}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta} e^{-i\sin\theta - in\theta}\,d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta + n\theta)]\,d\theta - \frac{i}{2\pi}\int_0^{2\pi} e^{\cos\theta}[\sin(\sin\theta + n\theta)]\,d\theta \end{aligned}$$

Let $g(\theta) = e^{\cos\theta}[\sin(\sin\theta + n\theta)]$, then $g(2\pi - \theta) = -e^{\cos\theta}[\sin(\sin\theta + n\theta)] = -g(\theta)$. Thus $\displaystyle\int_0^{2\pi} e^{\cos\theta}[\sin(\sin\theta + n\theta)]\,d\theta = 0$.

Thus $a_n = \displaystyle\frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta + n\theta)]\,d\theta$.

In particular, $a_{-n} = \displaystyle\frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta - n\theta)]\,d\theta$ for $n = 1, 2, \ldots$.

But we know that $e^{\frac{1}{z}} = 1 + \displaystyle\sum_{n=1}^{\infty} \frac{1}{n!z^n}$.

Therefore, comparing the two expansions we get for $n = 1, 2, \ldots$,

$$\frac{1}{2\pi}\int_0^{2\pi} e^{\cos\theta}[\cos(\sin\theta - n\theta)]\,d\theta = \frac{1}{n!}$$

Since $e^{\cos 2\pi - \theta}\cos(\sin(2\pi - \theta) - n(2\pi - \theta)) = e^{\cos\theta}[\cos(\sin\theta - n\theta)]$, we can double the integral and halve the limit to obtain

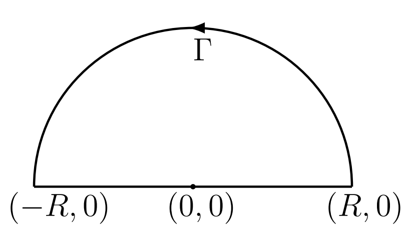
$$\frac{1}{\pi}\int_0^{\pi} e^{\cos\theta}\cos(\sin\theta - n\theta)\,d\theta = \frac{1}{n!}$$

∎

Question 2(b) *Show that* $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{1 + x^4} = \frac{\pi}{\sqrt{2}}$.

Solution.

We take $f(z) = \frac{1}{1+z^4}$ and the contour γ consisting of Γ a semicircle of radius R with center $(0, 0)$ lying in the upper half plane, and the line joining $(-R, 0)$ and $(R, 0)$.



By Cauchy's residue theorem $\int_\gamma \frac{dz}{1+z^4} = 2\pi i$(sum of residues at poles of $f(z)$ in the upper half plane).

Clearly $\frac{1}{1+z^4}$ has two simple poles at $z = e^{\frac{\pi i}{4}}$ and $z = e^{\frac{3\pi i}{4}}$ inside the contour.

Residue at $z = e^{\frac{\pi i}{4}}$ is $\frac{1}{\frac{d(z^4+1)}{dz}} = \frac{1}{4e^{\frac{3\pi i}{4}}}$.

Residue at $z = e^{\frac{3\pi i}{4}}$ is $\frac{1}{4e^{\frac{9\pi i}{4}}} = \frac{1}{4e^{\frac{\pi i}{4}}}$.

$$\begin{aligned}
\text{Sum of residues} &= \frac{1}{4}[\cos\frac{3\pi}{4} - i\sin\frac{3\pi}{4} + \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}] \\
&= \frac{1}{4}[-\cos\frac{\pi}{4} - i\sin\frac{\pi}{4} + \cos\frac{\pi}{4} - i\sin\frac{\pi}{4}] \\
&= -\frac{i}{4}\frac{2}{\sqrt{2}} = -\frac{i}{2\sqrt{2}}
\end{aligned}$$

Thus $\lim_{R\to\infty} \int_\gamma \frac{dz}{1+z^4} = 2\pi i \frac{-i}{2\sqrt{2}} = \frac{\pi}{\sqrt{2}}$.

Now

$$\left| \int_\Gamma \frac{dz}{1+z^4} \right| \leq \int_0^\pi \frac{R}{R^4-1}\, d\theta = \frac{\pi R}{R^4-1}$$

on putting $z = Re^{i\theta}$ and using $|z^4+1| \geq R^4 - 1$ on Γ.

Thus $\int_\Gamma \frac{dz}{1+z^4} \to 0$ as $R \to \infty$. Consequently,

$$\lim_{R\to\infty} \int_\gamma \frac{dz}{1+z^4} = \int_{-\infty}^{\infty} \frac{dx}{1+x^4} = \frac{\pi}{\sqrt{2}}$$

as required. ∎