UPSC Civil Services Main 2002 - Mathematics Complex Analysis

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Question 1(a) *Suppose that f and g are two analytic functions on the set $\mathbb{C}$ of all complex numbers with $f(\frac{1}{n}) = g(\frac{1}{n})$ for $n = 1, 2, 3, \ldots$, then show that $f(z) = g(z)$ for all $z \in \mathbb{C}$.*

Solution. Let $G(z) = f(z) - g(z)$, then $G(\frac{1}{n}) = 0$ for $n = 1, 2, \ldots$. We shall show that $G(z) \equiv 0$ for $z \in \mathbb{C}$ which would prove the result.

Let $G(z) = \sum_{n=0}^{\infty} a_n z^n$ be the power series of $G(z)$ with center 0 and radius of convergence R, clearly $R > 0$. We shall now prove that $a_n = 0$ for every n.

If $a_n \neq 0$ for some n, let a_k be the first non-zero coefficient. Then

$$G(z) = z^k(a_k + a_{k+1}z + \ldots) = z^k H(z)$$

Clearly $H(z)$ is analytic in $|z| < R$, and $H(0) \neq 0$. We now claim that $H(z) \neq 0$ in a neighborhood $|z| < \delta$ of 0. Let $\epsilon = \frac{|H(0)|}{2}$, then continuity of $H(z)$ at $z = 0$ implies that there exists a $\delta > 0$ such that $|z| < \delta \Rightarrow |H(z) - H(0)| < \epsilon$ or $|H(0)| - \epsilon < |H(z)| < |H(0)| + \epsilon$ for $|z| < \delta$. Thus $|H(z)| > \frac{|H(0)|}{2} > 0$ for $|z| < \delta$. Consequently, $G(z) \neq 0$ for any z in $0 < |z| < \delta$. But this is not possible, as $|z| < \delta$ contains all but finitely many $\frac{1}{n}$, at which $G(z)$ vanishes. Thus our assumption that $a_n \neq 0$ for some n is false, thus $G(z) \equiv 0$ in $|z| < R$.

Let z' be any point in $\mathbb{C}$, and let $r(t), a \leq t \leq b$ be a continuous curve joining 0 and z'. Using uniform continuity of $r(t)$, we get a partition $a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$ such that $r(t_0) = 0, r(t_1) = z_1, \ldots, r(t_n) = r(b) = z'$, and $|z_j - z_{j-1}| < R$.

Now the disc $K_0 = |z - 0| < R$ contains z_1, the center of disc $K_1 = |z - z_1| < R$. Since $G(z_1) = 0$ as $z_1 \in K_0 \cap K_1$, and $K_0 \cap K_1$ contains a sequence of points y_n such that $y_n \to z_1$ and $G(y_n) = 0$, we can prove as before that $G(z) \equiv 0$ in K_1. Proceeding in this way, in n steps we get $G(z) \equiv 0$ in K_n, or $G(z') = 0$. Since z' is an arbitrary point of $\mathbb{C}$, we get $G(z) \equiv 0$ in $\mathbb{C}$. ∎

Question 2(a) *Show that when $0 < |z-1| < 2$, the function $f(z) = \dfrac{z}{(z-1)(z-3)}$ has the Laurent series expansion in powers of $(z-1)$ as*

$$\frac{-1}{2(z-1)} - 3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}$$

Solution. Let $\zeta = z-1$, so that

$$f(z) = \frac{z}{(z-1)(z-3)} = \frac{\zeta+1}{\zeta(\zeta-2)} = -\frac{1}{2\zeta} + \frac{3}{2(\zeta-2)}$$

Now for $0 < |\zeta| < 2$, $\dfrac{3}{2(\zeta-2)} = \dfrac{3}{2}\cdot\dfrac{-1}{2}\cdot\Big(1-\dfrac{\zeta}{2}\Big)^{-1}$ and $\Big|\dfrac{\zeta}{2}\Big| < 1$. Consequently,

$$\frac{3}{2(\zeta-2)} = -\frac{3}{4}\sum_{n=0}^{\infty}\Big(\frac{\zeta}{2}\Big)^n$$

and

$$f(z) = -\frac{1}{2\zeta} - \frac{3}{4}\sum_{n=0}^{\infty}\Big(\frac{\zeta}{2}\Big)^n = -\frac{1}{2(z-1)} - 3\sum_{n=0}^{\infty}\frac{(z-1)^n}{2^{n+2}}$$

which is the desired Laurent series expansion. ∎

Question 2(b) *Establish by contour integration*

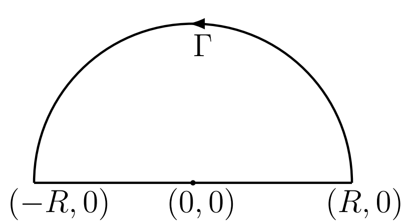
$$\int_0^{\infty}\frac{\cos(ax)}{x^2+1}\,dx = \frac{\pi}{2}e^{-a},\ \textit{where } a \geq 0$$

Solution. Let I be the given integral. Put $ax = t$, so that

$$I = \int_0^{\infty}\frac{\cos t}{\frac{t^2}{a^2}+1}\,\frac{dt}{a} = a\int_0^{\infty}\frac{\cos t}{t^2+a^2}\,dt$$

We shall now prove that $\displaystyle\int_0^{\infty}\frac{\cos t}{t^2+a^2}\,dt = \frac{\pi}{2a}e^{-a}$, which will show that $I = \frac{\pi}{2}e^{-a}$ as required.

Clearly $\dfrac{\cos x}{x^2+a^2}$ is the real part of $\dfrac{e^{ix}}{x^2+a^2}$. We consider the integral $\int_\gamma f(z)\,dz$ where $f(z) = \dfrac{e^{iz}}{z^2+a^2}$ and γ is the contour consisting of the line joining $(-R,0)$ and $(R,0)$ and Γ, which is the arc of the circle of radius R and center $(0,0)$ lying in the upper half plane.



Clearly on Γ, if we put $z = Re^{i\theta}$, then $0 \le \theta \le \pi$ and

$$\left|\int_\Gamma \frac{e^{iz}\,dz}{z^2+a^2}\right| = \left|\int_0^\pi \frac{Rie^{i\theta}e^{iRe^{i\theta}}\,d\theta}{R^2e^{2i\theta}+a^2}\right| \le \int_0^\pi \left|\frac{Rie^{i\theta}e^{iRe^{i\theta}}}{R^2e^{2i\theta}+a^2}\right| d\theta$$

But $|e^{iRe^{i\theta}}| = |e^{iR\cos\theta}e^{-R\sin\theta}| = e^{-R\sin\theta} \le 1$ as $\sin\theta \ge 0$ for $0 \le \theta \le \pi$. $|z^2+a^2| \ge |z|^2 - a^2 = R^2 - a^2$. Therefore

$$\left|\int_\Gamma \frac{e^{iz}\,dz}{z^2+a^2}\right| \le \int_0^\pi \frac{R}{R^2-a^2}\,d\theta = \frac{\pi R}{R^2-a^2}$$

Hence $\displaystyle\int_\Gamma \frac{e^{iz}\,dz}{z^2+a^2} \to 0$ as $R \to \infty$.

Now $\displaystyle\int_\gamma \frac{e^{iz}\,dz}{z^2+a^2} = 2\pi i(\text{sum of residues at poles inside } \gamma)$.

But the only pole in the upper half plane is $z = ia, (a > 0)$ and the residue at $z = ia$ is $\dfrac{e^{i(ia)}}{2ia} = \dfrac{e^{-a}}{2ia}$. Thus

$$\begin{aligned}
\lim_{R\to\infty}\int_\gamma \frac{e^{iz}\,dz}{z^2+a^2} &= \int_{-\infty}^{\infty} \frac{e^{ix}\,dx}{x^2+a^2} = 2\pi i \cdot \frac{e^{-a}}{2ia} = \frac{\pi e^{-a}}{a} \\
\Longrightarrow \int_{-\infty}^{\infty} \frac{\cos x\,dx}{x^2+a^2} &= \frac{\pi e^{-a}}{a}, \int_{-\infty}^{\infty} \frac{\sin x\,dx}{x^2+a^2} = 0 \\
\Longrightarrow \int_0^{\infty} \frac{\cos x\,dx}{x^2+a^2} &= \frac{\pi e^{-a}}{2a} \qquad \because \cos x = \cos(-x)
\end{aligned}$$

This completes the proof. ∎