UPSC Civil Services Main 2003 - Mathematics Complex Analysis

Brij Bhooshan

Asst. Professor

B.S.A. College of Engg & Technology

Mathura

Question 1(a) *Determine all the bilinear transformations which transform the unit circle* $|z| \leq 1$ *into the unit circle* $|w| \leq 1$.

Solution. Let the required transformation be $w = \dfrac{az+b}{cz+d}$. Clearly $z = -\frac{b}{a} \Rightarrow w = 0$ and $z = -\frac{d}{c} \Rightarrow w = \infty$. Since $0, \infty$ are inverse points with respect to the circle $|w| = 1$, then $-\frac{b}{a}, -\frac{d}{c}$ are inverse points with respect to the circle $|z| = 1$ (note that R, S different from 0 are said to be inverse points with respect to $|z| = 1$ if O, R, S are collinear and $OR \cdot OS = 1$). Thus if we set $-\frac{b}{a} = \alpha$, then $-\frac{d}{c} = \frac{1}{\overline{\alpha}}$ and we get

$$w = \frac{a}{c}\,\frac{z-\alpha}{z-\frac{1}{\overline{\alpha}}} = \frac{a\overline{\alpha}}{c}\,\frac{z-\alpha}{\overline{\alpha}z-1}$$

Since $|z| = 1$ maps onto $|w| = 1$, we take $z = 1$ to get $\left|\dfrac{a\overline{\alpha}}{c}\,\dfrac{1-\alpha}{\overline{\alpha}-1}\right| = 1$. But $|1-\alpha| = |1-\overline{\alpha}|$, therefore $\left|\dfrac{a\overline{\alpha}}{c}\right| = 1$. Let $\dfrac{a\overline{\alpha}}{c} = e^{i\theta}, \theta \in \mathbb{R}$, so that

$$w = e^{i\theta}\frac{z-\alpha}{\overline{\alpha}z-1}$$

We now check that when $|z| = 1$, we have $|w| = 1$.

$$\begin{aligned} |w| &= |e^{i\theta}|\left|\frac{z-\alpha}{\overline{\alpha}z-1}\right| \\ &= |\overline{z}|\left|\frac{z-\alpha}{\overline{\alpha}-\overline{z}}\right| \quad (\because z\overline{z} = 1) \\ &= 1 \quad (\because |z-\alpha| = |\overline{\alpha}-\overline{z}|) \end{aligned}$$

Now let $|z| < 1$. Then

$$\begin{aligned}
w\overline{w} - 1 &= e^{i\theta}\frac{z-\alpha}{\overline{\alpha}z-1} \cdot e^{-i\theta}\frac{\overline{z}-\overline{\alpha}}{\alpha\overline{z}-1} - 1 \\
&= \frac{z\overline{z} - \overline{\alpha}z - \alpha\overline{z} + \alpha\overline{\alpha}}{(\overline{\alpha}z-1)(\alpha\overline{z}-1)} - 1 \\
&= \frac{z\overline{z} - \overline{\alpha}z - \alpha\overline{z} + \alpha\overline{\alpha} - \alpha\overline{\alpha}z\overline{z} + \overline{\alpha}z + \alpha\overline{z} - 1}{(\overline{\alpha}z-1)(\alpha\overline{z}-1)} \\
&= \frac{z\overline{z} + \alpha\overline{\alpha} - \alpha\overline{\alpha}z\overline{z} - 1}{|\overline{\alpha}z-1|^2} \\
&= \frac{(z\overline{z}-1)(1-\alpha\overline{\alpha})}{|\overline{\alpha}z-1|^2}
\end{aligned}$$

Thus if $|\alpha| < 1$, then $|w| < 1$. This shows that the transformation

$$w = e^{i\theta}\frac{z-\alpha}{\overline{\alpha}z-1}, \theta \in \mathbb{R}, |\alpha| < 1$$

maps the interior of $|z| = 1$ onto the interior of $|w| = 1$ and the boundary of $|z| = 1$ onto the boundary of $|w| = 1$. Thus all bilinear transforms which map $|z| \le 1$ onto $|w| \le 1$ are given by

$$w = e^{i\theta}\frac{z-\alpha}{\overline{\alpha}z-1}, \theta \in \mathbb{R}, |\alpha| < 1$$

Note: If $|\alpha| > 1$, then the interior of $|z| = 1$ would map onto the exterior of $|w| = 1$. The boundary will map onto the boundary, as before. ∎

Question 2(a) 1. *Discuss the transformation* $W = \left(\frac{z-ic}{z+ic}\right)^2$, *c real, showing that the upper half of the W-plane corresponds to the interior of a semicircle lying to the right of the imaginary axis in the z-plane.*

2. *Using the method of contour integration prove that*

$$\int_0^{\pi} \frac{a\,d\theta}{a^2 + \sin^2\theta} = \frac{\pi}{\sqrt{1+a^2}} \qquad (a > 0)$$

Solution.

1. We need to assume $c > 0$ as otherwise the question is incorrect.

Let $W = U + iV$, so that

$$\begin{aligned}
U + iV &= \left(\frac{x + i(y-c)}{x + i(y+c)}\right)^2 \\
&= \left(\frac{(x + i(y-c))(x - i(y+c))}{x^2 + (y+c)^2}\right)^2 \\
&= \left(\frac{x^2 + y^2 - c^2 - 2icx}{x^2 + (y+c)^2}\right)^2 \\
&= \frac{(x^2 + y^2 - c^2)^2 - 4c^2x^2 - 4icx(x^2 + y^2 - c^2)}{[x^2 + (y+c)^2]^2} \\
\Rightarrow U &= \frac{(x^2 + y^2 - c^2)^2 - 4c^2x^2}{[x^2 + (y+c)^2]^2} \\
V &= \frac{-4cx(x^2 + y^2 - c^2)}{[x^2 + (y+c)^2]^2} = \frac{4cx(c^2 - x^2 - y^2)}{[x^2 + (y+c)^2]^2}
\end{aligned}$$

Thus if z belongs to the interior of the semicircle given by $x^2 + y^2 = c^2, x \geq 0$, then $V > 0$, which means that $U + iV$ is in the upper half plane.

For any point on the line $x = 0$, we have $V = 0$ and $U = \dfrac{(y^2 - c^2)^2}{(y+c)^4} = \left(\dfrac{y-c}{y+c}\right)^2$. Clearly when y changes from $-c$ to c, U changes from ∞ to 0.

As z moves over the circle $x^2 + y^2 = c^2$, we have $V = 0$ and

$$U = \frac{-4c^2x^2}{(x^2 + (y+c)^2)^2} = \frac{-4c^2x^2}{(x^2 + y^2 + c^2 + 2yc)^2} = \frac{-4c^2x^2}{(2c^2 + 2yc)^2} = \frac{-x^2}{(y+c)^2} = -\frac{c^2 - y^2}{(y+c)^2} = -\frac{c-y}{c+y}$$

Let $y = c\cos\theta$, then $U = -\dfrac{1 - \cos\theta}{1 + \cos\theta} = -\tan^2\frac{\theta}{2}$. When y moves from $-c$ to c, i.e.z traverses the boundary of the semicircle, θ varies from π to 0, and U varies from $-\infty$ to 0. Thus the boundary of the semicircle $x^2 + y^2 = c^2$ with $x \geq 0$ is mapped onto the U-axis. Hence the semicircle $x^2 + y^2 = c^2$ with $x \geq 0$ is mapped onto $W = U + iV$ with $V \geq 0$.

2. Let the given integral be I. Then

$$I = \int_0^\pi \frac{a\,d\theta}{a^2 + \sin^2\theta} = \int_0^\pi \frac{2a\,d\theta}{2a^2 + (1 - \cos 2\theta)} = \int_0^{2\pi} \frac{a\,d\phi}{2a^2 + 1 - \cos\phi}$$

on putting $2\theta = \phi$. We now let $z = e^{i\phi}$ to obtain

$$I = \int\limits_{|z|=1} \frac{a\,dz}{iz(2a^2 + 1 - \frac{1}{2}(z + \frac{1}{z}))} = \frac{1}{i}\int\limits_{|z|=1} \frac{2a\,dz}{2(2a^2+1)z - (z^2+1)} = \int\limits_{|z|=1} \frac{2ai\,dz}{z^2 - 2(2a^2+1)z + 1}$$

Now $z^2 - 2(2a^2+1)z + 1 = 0 \Rightarrow z = 2a^2+1 \pm \sqrt{(2a^2+1)^2 - 1} = 2a^2+1 \pm 2a\sqrt{a^2+1}$.

Clearly $|2a^2+1+2a\sqrt{a^2+1}| > 1$ showing that $|2a^2+1-2a\sqrt{a^2+1}| < 1$ because the product of the roots is 1. Thus the only pole inside $|z| = 1$ is $z = 2a^2+1-2a\sqrt{a^2+1}$.

Residue at $z = 2a^2+1-2a\sqrt{a^2+1}$ is $\dfrac{1}{(2a^2+1-2a\sqrt{a^2+1}) - (2a^2+1+2a\sqrt{a^2+1})} = \dfrac{1}{-4a\sqrt{a^2+1}}$.

Thus $I = 2ai \cdot 2\pi i \cdot \dfrac{1}{-4a\sqrt{a^2+1}} = \dfrac{\pi}{\sqrt{a^2+1}}$.

∎