UPSC Civil Services Main 2004 - Mathematics Complex Analysis

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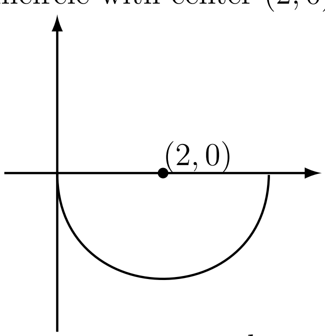
Question 1(a) *Find the image of the line $y = x$ under the mapping $w = \dfrac{4}{z^2+1}$ and draw it. Find the points where this transformation ceases to be conformal.*

Solution. Let $z = x + iy$. Then

$$\begin{aligned} w &= \frac{4}{z^2+1} = \frac{4}{x^2-y^2+1+2ixy} \\ &= \frac{4}{(x^2-y^2+1)^2+4x^2y^2}[x^2-y^2+1-2ixy] \end{aligned}$$

So if $x = y$, $w = \dfrac{4-8ix^2}{1+4x^4}$. Let $u = \dfrac{4}{1+4x^4}, v = \dfrac{-8x^2}{1+4x^4} \Rightarrow u^2+v^2 = 16\dfrac{1}{1+4x^4} = 4u \Rightarrow (u-2)^2+v^2 = 4, v \leq 0$. So the image of the line $x = y$ under the mapping $w = \dfrac{1}{z^2+1}$ is a semicircle with center $(2, 0)$, radius 2 and below the x-axis.



Conformality: $\dfrac{dw}{dz} = -\dfrac{8z}{(z^2+1)^2}$ when $z \neq \pm i$. Clearly $\dfrac{dw}{dz} \neq 0$ when $z \neq 0$. Thus the mapping is conformal at all points which are different from $z = 0, \pm i$ (as $\dfrac{dw}{dz}$ does not exist at $\pm i$). ∎

Question 2(a) *If all zeros of a polynomial $P(z)$ lie in a half plane, then show that zeros of the derivative $P'(z)$ also lie in the same half plane.*

Solution. We can assume without loss of generality that the zeros of $P(z)$ lie in the half plane $\operatorname{Re} z < 0$. Let $P(z) = \prod_{j=1}^{n}(z - \alpha_j)$ where $\alpha_j = x_j + iy_j, x_j < 0$.

If $\operatorname{Re} z \geq 0$, then $P(z) \neq 0$ and

$$\begin{aligned} \frac{P'(z)}{P(z)} &= \sum_{j=1}^{n} \frac{1}{z - \alpha_j} \\ &= \sum_{j=1}^{n} \frac{1}{x - x_j + i(y - y_j)} \\ &= \sum_{j=1}^{n} \frac{x - x_j - i(y - y_j)}{(x - x_j)^2 + (y - y_j)^2} \end{aligned}$$

Since $x_j < 0, 1 \leq j \leq n$, it follows that

$$\operatorname{Re}\left(\frac{P'(z)}{P(z)}\right) = \sum_{j=1}^{n} \frac{x - x_j}{(x - x_j)^2 + (y - y_j)^2} > 0$$

whenever $\operatorname{Re} z = x \geq 0$. Thus $\dfrac{P'(z)}{P(z)}$ and therefore $P'(z)$ has no zeros in the right half plane $\operatorname{Re} z \geq 0$. Hence all zeros of $P'(z)$ lie in the same half plane in which the zeros of $P(z)$ lie. ∎

Question 2(b) *Using Contour integration, evaluate*

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{1 - 2p\cos 2\theta + p^2}\, d\theta, \quad 0 < p < 1$$

Solution. Clearly

$$\int_0^{2\pi} \frac{\cos^2 3\theta}{1 - 2p\cos 2\theta + p^2}\, d\theta = \frac{1}{2}\int_0^{2\pi} \frac{1 + \cos 6\theta}{1 - 2p\cos 2\theta + p^2}\, d\theta$$

The integrand is the real part of $\dfrac{1 + e^{i6\theta}}{1 + 2p\cos 2\theta + p^2}$. Put $z = e^{i\theta}$, $dz = ie^{i\theta}\, d\theta$ or $d\theta = \dfrac{dz}{iz}$.

$$\begin{aligned} \int_0^{2\pi} \frac{1 + e^{i6\theta}}{1 + 2p\cos 2\theta + p^2}\, d\theta &= \frac{1}{i}\int_{|z|=1} \frac{1 + z^6}{1 - p(z^2 + \frac{1}{z^2}) + p^2}\, \frac{dz}{z} \\ &= \frac{1}{i}\int_{|z|=1} \frac{z(1 + z^6)}{-pz^4 + z^2(1 + p^2) - p}\, dz \\ &= \frac{1}{i}\int_{|z|=1} \frac{z(1 + z^6)}{(1 - pz^2)(z^2 - p)}\, dz \end{aligned}$$

Now the integrand has simple poles at $z = \pm\sqrt{p}, \pm\frac{1}{\sqrt{p}}$. Since $0 < p < 1$, the only poles inside $|z| = 1$ are $z = \pm\sqrt{p}$. The residue at $z = \sqrt{p}$ is

$$\lim_{z\to\sqrt{p}} \frac{(z-\sqrt{p})z(1+z^6)}{(1-pz^2)(z^2-p)} = \frac{\sqrt{p}(1+p^3)}{(1-p^2)2\sqrt{p}} = \frac{1+p^3}{2(1-p^2)}$$

Similarly residue at $z = -\sqrt{p}$ is

$$\lim_{z\to-\sqrt{p}} \frac{(z+\sqrt{p})z(1+z^6)}{(1-pz^2)(z^2-p)} = \frac{-\sqrt{p}(1+p^3)}{(1-p^2)(-2\sqrt{p})} = \frac{1+p^3}{2(1-p^2)}$$

Thus

$$\begin{aligned}
\int_0^{2\pi} \frac{\cos^2 3\theta}{1-2p\cos 2\theta+p^2}\,d\theta &= \frac{1}{2}\,\mathrm{Re}\left(\int_0^{2\pi} \frac{1+e^{i6\theta}}{1+2p\cos 2\theta+p^2}\,d\theta\right) \\
&= \mathrm{Re}\left(\frac{1}{2i}2\pi i[\text{Sum of residues at } z = \pm\sqrt{p}]\right) \\
&= \pi\frac{1+p^3}{1-p^2} = \pi\frac{1-p+p^2}{1-p}
\end{aligned}$$

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