UPSC Civil Services Main 2005 - Mathematics Complex Analysis

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Paper II

Question 1(a) *If $f(z) = u + iv$ is an analytic function of the complex variable z and $u - v = e^x(\cos y - \sin y)$, determine $f(z)$ in terms of z.*

Solution. Let $F(z) = (1+i)f(z) = (1+i)(u+iv) = (u-v) + i(u+v)$. Now

$$\frac{\partial^2(u-v)}{\partial x^2} + \frac{\partial^2(u-v)}{\partial y^2} = e^x(\cos y - \sin y) + e^x(-\cos y + \sin y) = 0$$

Let $F(z) = U + iV$, where $U = u - v$ is harmonic. If $f(z)$ is analytic, then so is $F(z)$ and

$$\frac{dF}{dz} = \frac{\partial U}{\partial x} + i\frac{\partial V}{\partial x} = \frac{\partial U}{\partial x} - i\frac{\partial U}{\partial y}$$

as $\frac{\partial V}{\partial x} = -\frac{\partial U}{\partial y}$ by the Cauchy-Riemann equations. Thus

$$\begin{aligned} F'(z) &= e^x(\cos y - \sin y) - ie^x(-\sin y - \cos y) \\ &= e^x(\cos y + i\sin y) + ie^x(\cos y + i\sin y) \\ &= (1+i)e^x \cdot e^{iy} = (1+i)e^z \end{aligned}$$

Thus $F(z) = (1+i)e^z$ and $f(z) = e^z$, which is the required function. ∎

Question 2(a) *Expand* $f(z) = \dfrac{1}{(z+1)(z+3)}$ *in Laurent's series which is valid for (i)* $1 < |z| < 3$ *(ii)* $|z| > 3$ *and (iii)* $|z| < 1$.

Solution. Clearly $f(z) = \dfrac{1}{2}\Big(\dfrac{1}{z+1} - \dfrac{1}{z+3}\Big)$

(i) $1 < |z| < 3$. In this region

$$f(z) = \frac{1}{2}\Big[\frac{1}{z}\Big(1 + \frac{1}{z}\Big)^{-1} - \frac{1}{3}\Big(1 + \frac{z}{3}\Big)^{-1}\Big]$$

Since $|\frac{1}{z}| < 1$ and $|\frac{z}{3}| < 1$, we get

$$\begin{aligned} f(z) &= \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n \frac{1}{z^n} - \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\Big(\frac{z}{3}\Big)^n \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2}\frac{1}{z^{n+1}} - \sum_{n=0}^{\infty}\frac{(-1)^n}{2}\frac{z^n}{3^{n+1}} \end{aligned}$$

as Laurent's expansion in the region $1 < |z| < 3$.

(ii) $|z| > 3$. In this region

$$f(z) = \frac{1}{2z}\Big[\Big(1 + \frac{1}{z}\Big)^{-1} - \Big(1 + \frac{3}{z}\Big)^{-1}\Big]$$

Now $|\frac{1}{z}| < 1$ and $|\frac{3}{z}| < 1$, so we get

$$\begin{aligned} f(z) &= \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n \frac{1}{z^n} - \frac{1}{2z}\sum_{n=0}^{\infty}(-1)^n\frac{3^n}{z^n} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2}\frac{1-3^n}{z^{n+1}} \end{aligned}$$

as Laurent's expansion in the region $|z| > 3$. This is Taylor's expansion of $f(z)$ around ∞.

(iii) $|z| < 1$. In this region

$$f(z) = \frac{1}{2}\Big[(1+z)^{-1} - \frac{1}{3}\Big(1 + \frac{z}{3}\Big)^{-1}\Big]$$

Now $|z| < 1, |\frac{z}{3}| < 1$, so we get

$$\begin{aligned} f(z) &= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n z^n - \frac{1}{6}\sum_{n=0}^{\infty}(-1)^n\frac{z^n}{3^n} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{2}\Big(1 - \frac{1}{3^{n+1}}\Big)z^n \end{aligned}$$

as Laurent's expansion valid in $|z| < 1$. This has no negative powers of z as $f(z)$ is analytic in $|z| < 1$. ∎