UPSC Civil Services Main 2006 - Mathematics Complex Analysis

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Question 1(a) *Determine all the bilinear transformations which map the half plane* $\text{Im}(z) \geq 0$ *into the unit circle* $|w| \leq 1$.

Solution. Since the x-axis is to be mapped onto the circle $|w| = 1$, we determine conditions on a, b, c, d where

$$w = \frac{az + b}{cz + d}$$

is the desired bilinear transformation, such that points $z = 0, z = 1, z = \infty$ are mapped onto points with modulus 1.

First of all, $c \neq 0$, because if $c = 0$ then the image of $z = \infty$ would be $w = \infty$, which is not possible.

Clearly $z = \infty$ is mapped onto $w = \frac{a}{c}$, thus we must have $|\frac{a}{c}| = 1$ or $|a| = |c| \neq 0$.

When $z = 0$, $w = \frac{b}{d}$ (note that $d \neq 0$, because otherwise $z = 0$ would be mapped onto ∞), thus $|w| = 1$ gives us $|b| = |d| \neq 0$.

Since $a \neq 0, c \neq 0$, we can write $w = \dfrac{az + b}{cz + d} = \dfrac{a}{c}\left(\dfrac{z + \frac{b}{a}}{z + \frac{d}{c}}\right) = e^{i\alpha}\dfrac{z - z_0}{z - z_1}$ with $e^{i\alpha} = \dfrac{a}{c}, z_0 = -\dfrac{b}{a}, z_1 = -\dfrac{d}{c}$. But $\left|\dfrac{b}{a}\right| = \left|\dfrac{d}{c}\right|$, so $|z_0| = |z_1|$. Thus we have proved that w can be written in the form

$$w = e^{i\alpha}\left(\frac{z - z_0}{z - z_1}\right) \text{ with } \alpha \in \mathbb{R}, |z_0| = |z_1|$$

We now use the fact that the image of $z = 1$ has modulus 1, and get

$$\begin{aligned} |1 - z_0| &= |1 - z_1| \\ \Rightarrow (1 - z_0)(1 - \overline{z_0}) &= (1 - z_1)(1 - \overline{z_1}) \\ \Rightarrow 1 - z_0 - \overline{z_0} + z_0\overline{z_0} &= 1 - z_1 - \overline{z_1} + z_1\overline{z_1} \\ \Rightarrow z_0 + \overline{z_0} &= z_1 + \overline{z_1} \quad \because |z_0| = |z_1| \Rightarrow z_0\overline{z_0} = z_1\overline{z_1} \\ \Rightarrow \mathrm{Re}(z_0) &= \mathrm{Re}(z_1) \end{aligned}$$

This gives us $z_0 = z_1$ or $z_0 = \overline{z_1}$ because if $z_0 = x + iy_0, z_1 = x + iy_1$, then $x^2 + y_0^2 = x^2 + y_1^2 \Rightarrow y_0^2 = y_1^2 \Rightarrow y_1 = \pm y_0$. If $z_0 = z_1$, then $w = e^{i\alpha}$, a constant, which is not possible, therefore $z_1 = \overline{z_0}$ and the transformation w can be written as $w = e^{i\alpha}\left(\dfrac{z - z_0}{z - \overline{z_0}}\right)$.

This transformation maps z_0 to $w = 0$. Since we require the upper half plane $\mathrm{Im}(z) > 0$ to be mapped onto the interior of $|w| = 1$, we must have $\mathrm{Im}(z_0) > 0$. Thus any transformation which maps the real axis onto $|w| = 1$ and the region $\mathrm{Im}(z) > 0$ to the interior of $|w| = 1$ can be written in the form

$$w = e^{i\alpha}\left(\frac{z - z_0}{z - \overline{z_0}}\right), \quad \alpha \in \mathbb{R}, \mathrm{Im}(z_0) > 0$$

We now prove the converse — any bilinear transformation $w = e^{i\alpha}\left(\dfrac{z - z_0}{z - \overline{z_0}}\right), \quad \alpha \in \mathbb{R}, \mathrm{Im}(z_0) > 0$ maps $\mathrm{Im}(z) > 0$ to $|w| \leq 1$.

If z is such that $\mathrm{Im}(z) \geq 0$, then it can be seen easily that $|z - z_0| < |z - \overline{z_0}|$, therefore $|w| < 1$. Similarly if we assume that $\mathrm{Im}(z) < 0$, then $|z - z_0| > |z - \overline{z_0}|$ and therefore $|w| > 1$. Clearly when z lies on the real axis, then $|w| = 1$ as $|z - z_0| = |z - \overline{z_0}|$. This proves the result.

Hence all bilinear transformations of the required type are of the form

$$\left\{w = e^{i\alpha}\left(\frac{z - z_0}{z - \overline{z_0}}\right), \quad \alpha \in \mathbb{R}, \mathrm{Im}(z_0) > 0\right\}$$

Alternate Solution: Since the x-axis is mapped to the unit circle $|w| = 1$, the reflection of the image of z in $|w| = 1$ is the same as the image of the reflection of z in the x-axis i.e. $\overline{z}$. Thus

$$\begin{aligned} \overline{\left(1 \Big/ \frac{az + b}{cz + d}\right)} &= \frac{a\overline{z} + b}{c\overline{z} + d} \\ \Rightarrow \quad \overline{(cz + d)}(c\overline{z} + d) &= \overline{(az + b)}(a\overline{z} + b) \\ \Rightarrow \quad c\overline{c}\overline{z}^2 + (c\overline{d} + \overline{c}d)\overline{z} + d\overline{d} &= a\overline{a}\overline{z}^2 + (a\overline{b} + \overline{a}b)\overline{z} + b\overline{b} \end{aligned}$$

Comparing coefficients of the powers of $\overline{z}$ we have $|a| = |c|, |b| = |d|, \mathrm{Re}\,(b\overline{a}) = \mathrm{Re}\,(d\overline{c})$, thus since $ad - bc \neq 0$, we have $|a| = |c| \neq 0, |b| = |d| \neq 0$, and $\mathrm{Re}\,(\frac{b}{a}) = \mathrm{Re}\,(\frac{b\overline{a}}{a\overline{a}}) = \mathrm{Re}\,(\frac{d\overline{c}}{c\overline{c}}) = \mathrm{Re}\,(\frac{d}{c})$.

Thus we can write $w = \frac{az+b}{cz+d} = \frac{a}{c}\left(\frac{z+\frac{b}{a}}{z+\frac{d}{c}}\right) = e^{i\alpha}\frac{z-z_0}{z-z_1}$ with $e^{i\alpha} = \frac{a}{c}, z_0 = -\frac{b}{a}, z_1 = -\frac{d}{c}$. Thus $\operatorname{Re} z_0 = \operatorname{Re} z_1, |z_0| = |z_1|$, hence $z_0 = z_1$ or $z_0 = \overline{z_1}$. The former is not possible as it would make $ad - bc = 0$, hence $z_0 = \overline{z_1}$. For the same reason, $\operatorname{Im} z_0 \neq 0$. So we have

$$w = e^{i\alpha}\left(\frac{z-z_0}{z-\overline{z_0}}\right), \quad \alpha \in \mathbb{R}$$

Now in addition, since the upper half plane $\operatorname{Im}(z) > 0$ is mapped onto the interior of $|w| = 1$, and the image of z_0 is 0, and thus inside the unit circle, so z_0 is in the upper half plane, hence $\operatorname{Im}(z_0) > 0$.

Hence the required set of bilinear transformations is

$$\left\{w = e^{i\alpha}\left(\frac{z-z_0}{z-\overline{z_0}}\right), \quad \alpha \in \mathbb{R}, \operatorname{Im}(z_0) > 0\right\}$$

The converse is proved as above. ∎

Question 2(a) *With the aid of residues, evaluate*

$$\int_0^\pi \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\,d\theta, \quad -1 < a < 1$$

Solution. Let

$$\begin{aligned} I &= \int_0^\pi \frac{\cos 2\theta}{1 - 2a\cos\theta + a^2}\,d\theta \\ &= \int_0^\pi \frac{(\cos 2\theta)(1 + 2a\cos\theta + a^2)}{(1+a^2)^2 - 4a^2\cos^2\theta}\,d\theta \\ &= \int_0^\pi \frac{(\cos 2\theta)(1+a^2)}{(1+a^2)^2 - 4a^2\cos^2\theta}\,d\theta + \int_0^\pi \frac{2a\cos\theta\cos 2\theta}{(1+a^2)^2 - 4a^2\cos^2\theta}\,d\theta \end{aligned}$$

Since $\cos(\pi - \theta) = -\cos\theta$, on puuting $\pi - \theta = \alpha$ we get

$$\int_0^\pi \frac{2a\cos\theta\cos 2\theta}{(1+a^2)^2 - 4a^2\cos^2\theta}\,d\theta = \int_\pi^0 \frac{2a(-\cos\alpha)\cos 2\alpha}{(1+a^2)^2 - 4a^2\cos^2\alpha}\,(-d\alpha)$$

showing that $\int_0^\pi \frac{2a\cos\theta\cos 2\theta}{(1+a^2)^2 - 4a^2\cos^2\theta}\,d\theta = 0$. Thus

$$I = \int_0^\pi \frac{(\cos 2\theta)(1+a^2)}{(1+a^2)^2 - 2a^2(1+\cos 2\theta)}\,d\theta$$

Putting $2\theta = \beta$, we get

$$I = \frac{1}{2}\int_0^{2\pi} \frac{(\cos\beta)(1+a^2)}{(1+a^2)^2 - 2a^2(1+\cos\beta)}\,d\beta = \frac{1}{2}\int_0^{2\pi} \frac{(1+a^2)\cos\beta}{1 + a^4 - 2a^2\cos\beta}\,d\beta$$

We now put $z = e^{i\beta}$, so that $dz = iz\,d\beta$ or $d\beta = \dfrac{dz}{iz}$.

$$\begin{aligned}
I &= \frac{1}{2}\int_{|z|=1} \frac{(1+a^2)\frac{1}{2}(z+\frac{1}{z})}{1+a^4-a^2(z+\frac{1}{z})}\,\frac{dz}{iz} \\
&= \frac{1+a^2}{4i}\int_{|z|=1} \frac{z^2+1}{z\big[(1+a^4)z-a^2z^2-a^2\big]}\,dz \\
&= \frac{1+a^2}{4a^2 i}\int_{|z|=1} \frac{-(z^2+1)}{z\big[z^2-(a^2+\frac{1}{a^2})z+1\big]}\,dz \\
&= \frac{(1+a^2)i}{4a^2}\int_{|z|=1} \frac{z^2+1}{z(z-a^2)(z-\frac{1}{a^2})}\,dz
\end{aligned}$$

Clearly the integrand has simple poles at $z = 0, z = a^2, z = \frac{1}{a^2}$, out of which $z = 0$ and $z = a^2$ lie inside $|z| = 1$ as $-1 < a < 1$.

Residue at $z = 0$ is $\displaystyle\lim_{z\to 0} \frac{(z^2+1)z}{z(z-a^2)(z-\frac{1}{a^2})} = 1$.

Residue at $z = a^2$ is $\displaystyle\lim_{z\to 0} \frac{(z^2+1)(z-a^2)}{z(z-a^2)(z-\frac{1}{a^2})} = \frac{a^4+1}{a^4-1}$.

Cauchy's residue theorem (the integral around a curve $= 2\pi i\cdot$ sum of residues at poles inside the curve) now gives us

$$I = \frac{(1+a^2)i}{4a^2}\cdot 2\pi i\left[1+\frac{a^4+1}{a^4-1}\right] = -2\pi\frac{(1+a^2)\cdot 2a^4}{4a^2(a^4-1)} = \frac{\pi a^2}{1-a^2}$$

∎

Question 2(b) *Prove that all the roots of $z^7 - 5z^3 + 12 = 0$ lie between the circles $|z| = 1$ and $|z| = 2$.*

Solution. Let $g(z) = z^7 - 5z^3, f(z) = 12$, then

1. On $|z| = 1$, $|g(z)| \leq |z^7| + 5|z^3| = 6 < 12 = |f(z)|$.
2. Both $g(z)$ and $f(z)$ are analytic on and within $|z| < 1$
3. Both $g(z)$ and $f(z)$ have no zeros on $|z| = 1$

By Rouche's theorem, $f(z) + g(z) = z^7 - 5z^3 + 12$ and $f(z)$ have the same number of zeros inside $|z| = 1$. But $f(z) = 12$ has no zeros anywhere and in particular in the region $|z| < 1$, therefore $z^7 - 5z^3 + 12$ has no zeros inside the unit circle.

Now we take $g(z) = 12 - 5z^3$, $f(z) = z^7$.

1. On $|z| = 2$, $|g(z)| \leq 12 + 5|z^3| = 52 < 2^7 = |f(z)|$.

2. Both $g(z)$ and $f(z)$ are analytic on and within $|z| < 2$

Therefore by Rouche's theorem, $g(z) + f(z) = z^7 - 5z^3 + 12$ and $f(z)$ have the same number of zeros inside $|z| = 2$. Since $f(z) = z^7$ has 7 zeros ($z = 0$ is a zero of order 7 of $f(z)$) inside $|z| = 2$, the given polynomial has seven zeros inside $|z| = 2$ i.e. all its zeros lie inside $|z| = 2$.

Since $z^7 - 5z^3 + 12$ has no zeros inside and on $|z| = 1$, therefore all zeros lie in the ring $1 < |z| < 2$. ∎