UPSC Civil Services Main 1980 - Mathematics
Linear Algebra

Brij Bhooshan

Asst. Professor

B.S.A. College of Engg & Technology

Mathura

Question 1(a) *Define the rank of a matrix. Prove that a system of equations* $\mathbf{Ax} = \mathbf{b}$ *is consistent if and only if* $\operatorname{rank}(\mathbf{A}, \mathbf{b}) = \operatorname{rank}\mathbf{A}$, *where* $(\mathbf{A}, \mathbf{b})$ *is the augmented matrix of the system.*

Solution. See 1987 question 3(a). ∎

Question 1(b) *Verify the Cayley Hamilton Theorem for the matrix* $\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, *and hence find* $\mathbf{A}^{-1}$.

Solution. The Cayley Hamilton theorem is — Every matrix $\mathbf{A}$ satisfies its characteristic equation $|x\mathbf{I} - \mathbf{A}| = 0$. In the current problem, $|x\mathbf{I} - \mathbf{A}| = \begin{vmatrix} x-2 & -1 \\ -1 & x-2 \end{vmatrix} = x^2 - 4x + 4 - 1 = x^2 - 4x + 3$. Thus we need to show that $\mathbf{A}^2 - 4\mathbf{A} + 3\mathbf{I} = \mathbf{0}$. Now $\mathbf{A}^2 = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$, so $\mathbf{A}^2 - 4\mathbf{A} + 3\mathbf{I} = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix} - 4 \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + 3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, verifying the Cayley Hamilton Theorem.

$\mathbf{A}^2 - 4\mathbf{A} + 3\mathbf{I} = \mathbf{0} \Rightarrow \mathbf{A}(\mathbf{A} - 4\mathbf{I}) = -3\mathbf{I} \Rightarrow \mathbf{A}^{-1} = -\frac{1}{3}(\mathbf{A} - 4\mathbf{I}) = -\frac{1}{3}(\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} - 4 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}) = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$. ∎

Question 2(a) *Prove that if* $\mathbf{P}$ *is any non-singular matrix of order* n, *then the matrices* $\mathbf{P}^{-1}\mathbf{AP}$ *and* $\mathbf{A}$ *have the same characteristic polynomial.*

Solution. The characteristic polynomial of $\mathbf{P}^{-1}\mathbf{AP}$ is $|x\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}| = |x\mathbf{P}^{-1}\mathbf{P} - \mathbf{P}^{-1}\mathbf{AP}| = |\mathbf{P}^{-1}||x\mathbf{I} - \mathbf{A}||\mathbf{P}| = |x\mathbf{I} - \mathbf{A}|$ which is the characteristic polynomial of $\mathbf{A}$. ∎

Question 2(b) *Find the eigenvalues and eigenvectors of the matrix* $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$.

Solution. The characteristic equation of $\mathbf{A} = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$ is $\begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0 \Rightarrow -(9 - \lambda^2) - 16 = 0 \Rightarrow \lambda^2 - 25 = 0 \Rightarrow \lambda = 5, -5$.

If (x_1, x_2) is an eigenvector for $\lambda = 5$, then $\begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0} \Rightarrow 2x_1 - 4x_2 = 0 \Rightarrow x_1 = 2x_2$. Thus $(2x, x), x \in \mathbb{R}, x \neq 0$ gives all eigenvectors for $\lambda = 5$, in particular, we can take $(2, 1)$ as an eigenvector for $\lambda = 5$.

If (x_1, x_2) is an eigenvector for $\lambda = -5$, then $\begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0} \Rightarrow 4x_1 + 2x_2 = 0 \Rightarrow x_2 = -2x_1$. Thus $(x, -2x), x \in \mathbb{R}, x \neq 0$ gives all eigenvectors for $\lambda = -5$, in particular, we can take $(1, -2)$ as an eigenvector for $\lambda = -5$. ∎

Question 3(a) *Find a basis for the vector space* $\mathcal{V} = \{p(x) \mid p(x) = a_0 + a_1 x + a_2 x^2\}$ *and its dimension.*

Solution. Let $f_1 = 1, f_2 = x, f_3 = x^2$, then f_1, f_2, f_3 are linearly independent, because $\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 = 0 \Rightarrow \alpha_1 + \alpha_2 x + \alpha_3 x^2 = 0(\text{zero polynomial}) \Rightarrow \alpha_1 = \alpha_2 = \alpha_3 = 0$.

f_1, f_2, f_3 generate $\mathcal{V}$ because $p(x) = a_0 + a_1 x + a_2 x^2 = a_0 f_1 + a_1 f_2 + a_2 f_3$ for any $p(x) \in \mathcal{V}$. Thus $\{f_1, f_2, f_3\}$ is a basis for $\mathcal{V}$ and its dimension is 3. ∎

Question 3(b) *Find the values of the parameter* λ *for which the system of equations*

$$\begin{aligned} x + y + 4z &= 1 \\ x + 2y - 2z &= 1 \\ \lambda x + y + z &= 1 \end{aligned}$$

will have (i) unique solution (ii) no solution.

Solution. The system will have the unique solution given by $\begin{pmatrix} 1 & 1 & 4 \\ 1 & 2 & -2 \\ \lambda & 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ if $\begin{vmatrix} 1 & 1 & 4 \\ 1 & 2 & -2 \\ \lambda & 1 & 1 \end{vmatrix} = 1(2+2) + 4(1-2\lambda) - 1(1+2\lambda) \neq 0$. Thus $4 + 4 - 8\lambda - 1 - 2\lambda \neq 0 \Rightarrow \lambda \neq \frac{7}{10}$.

When $\lambda = \frac{7}{10}$, the system is

$$\begin{aligned} x + y + 4z &= 1 \\ x + 2y - 2z &= 1 \\ 7x + 10y + 10z &= 10 \end{aligned}$$

This system has no solution as it is inconsistent: $4(x+y+4z) + 3(x+2y-2z) = 7x+10y+10z = 7$, but the third equation says that $7x + 10y + 10z = 10$. Thus there is a unique solution if $\lambda \neq \frac{7}{10}$, and no solution if $\lambda = \frac{7}{10}$. ∎

Question 3(c) *If $\mathcal{V}$ is a finite dimensional vector space and $\mathcal{M}$ is a subspace of $\mathcal{V}$, then show that each vector $\mathbf{x} \in \mathcal{V}$ can be uniquely expressed as $\mathbf{x} = \mathbf{y} + \mathbf{z}$, where $\mathbf{y} \in \mathcal{M}$ and $\mathbf{z} \in \mathcal{M}^{\perp}$, the orthogonal complement of $\mathcal{M}$.*

Solution. Let $\mathbf{v_1}, \ldots, \mathbf{v_m}$ be any orthonormal basis of $\mathcal{M}$, where $m = \dim \mathcal{M}$. Given $\mathbf{x} \in \mathcal{V}$, let $\mathbf{y} = \sum_{i=1}^{m} \langle \mathbf{x}, \mathbf{v_i} \rangle \mathbf{v_i}$, and $\mathbf{z} = \mathbf{x} - \mathbf{y}$. Clearly $\mathbf{y} \in \mathcal{M}$, and $\mathbf{x} = \mathbf{y} + \mathbf{z}$. Now $\langle \mathbf{z}, \mathbf{v_i} \rangle = \langle \mathbf{x}, \mathbf{v_i} \rangle - \langle \mathbf{y}, \mathbf{v_i} \rangle = \langle \mathbf{x}, \mathbf{v_i} \rangle - \sum_{j=1}^{m} \langle \mathbf{x}, \mathbf{v_j} \rangle \langle \mathbf{v_j}, \mathbf{v_i} \rangle = \langle \mathbf{x}, \mathbf{v_i} \rangle - \langle \mathbf{x}, \mathbf{v_i} \rangle = 0$. So $\langle \mathbf{z}, \mathbf{v_i} \rangle = 0, i = 1, \ldots, m \Rightarrow \langle \mathbf{z}, \mathbf{m} \rangle = 0$ for every $\mathbf{m} \in \mathcal{M}$, so $\mathbf{z} \in \mathcal{M}^{\perp}$.

Now if $\mathbf{x} = \mathbf{y}' + \mathbf{z}'$, then $\mathbf{y} - \mathbf{y}' = \mathbf{z}' - \mathbf{z}$. But $\mathbf{y} - \mathbf{y}' \in \mathcal{M}, \mathbf{z}' - \mathbf{z} \in \mathcal{M}^{\perp}$, so $\langle \mathbf{y} - \mathbf{y}', \mathbf{z}' - \mathbf{z} \rangle = 0 \Rightarrow \langle \mathbf{y} - \mathbf{y}', \mathbf{y} - \mathbf{y}' \rangle = 0 \Rightarrow ||\mathbf{y} - \mathbf{y}'|| = 0 \Rightarrow \mathbf{y} - \mathbf{y}' = \mathbf{0} \Rightarrow \mathbf{z}' - \mathbf{z} = \mathbf{0}$. Thus $\mathbf{y} = \mathbf{y}', \mathbf{z} = \mathbf{z}'$ and the representation is unique. ∎

Question 3(d) *Find one characteristic value and corresponding characteristic vector for the operators T on $\mathbb{R}^3$ defined as*

1. *T is a reflection on the plane $x = z$.*
2. *T is a projection on the plane $z = 0$.*
3. *$T(x, y, z) = (3x + y + z, 2y + z, z)$.*

Solution.

1. $T(x, y, z) = (z, y, x)$ because the midpoint of (x, y, z) and (z, y, x) lies on the plane $x = z$. $T(1, 0, 0) = (0, 0, 1), T(0, 1, 0) = (0, 1, 0), T(0, 0, 1) = (1, 0, 0)$. Thus it is clear that 1 is an eigenvalue, and $(0, 1, 0)$ is a corresponding eigenvector.
2. $T(1, 0, 0) = (1, 0, 0), T(0, 1, 0) = (0, 1, 0), T(0, 0, 1) = (0, 0, 0)$. Clearly 1 is an eigenvalue with $(1, 0, 0)$ or $(0, 1, 0)$ as eigenvectors.
3. $T(1, 0, 0) = (3, 0, 0), T(0, 1, 0) = (1, 2, 0), T(0, 0, 1) = (1, 1, 1)$. Clearly $(1, 0, 0)$ is an eigenvector, corresponding to the eigenvalue 3.

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