# UPSC Civil Services Main 1982 - Mathematics
# Linear Algebra

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**Question 1(a)** *Let $\mathcal{V}$ be a vector space. If $\dim \mathcal{V} = n$ with $n > 0$, prove that*

1. *any set of $n$ linearly independent vectors is a basis of $\mathcal{V}$.*
2. *$\mathcal{V}$ cannot be generated by fewer than $n$ vectors.*

**Solution.** From 1983 question 1(a) we get that any two bases of $\mathcal{V}$ have $n$ elements.

1. Let $\mathbf{v_1}, \ldots, \mathbf{v_n}$ be $n$ linearly independent vectors in $\mathcal{V}$. Then $\mathbf{v_1}, \ldots, \mathbf{v_n}$ generate $\mathcal{V}$ — if $\mathbf{v} \in \mathcal{V}$ is such that $\mathbf{v}$ is not a linear combination of $\mathbf{v_1}, \ldots, \mathbf{v_n}$, then $\mathbf{v}, \mathbf{v_1}, \ldots, \mathbf{v_n}$ are linearly independent, so $\dim \mathcal{V} > n$ which is not true. Thus $\mathbf{v_1}, \ldots, \mathbf{v_n}$ is a basis of $\mathcal{V}$ — here we have used the technique used to complete any linearly independent set to a basis.
2. $\mathcal{V}$ cannot be generated by fewer than $n$ vectors, because then it will have a basis consisting of less than $n$ elements, which contradicts the fact that $\dim \mathcal{V} = n$.

$\blacksquare$

**Question 1(b)** *Define a linear transformation. Prove that both the range and the kernel of a linear transformation are vector spaces.*

**Solution.** Let $\mathcal{V}$ and $\mathcal{W}$ be two vector spaces. A mapping $\mathbf{T} : \mathcal{V} \longrightarrow \mathcal{W}$ is said to be a linear transformation if

1. $\mathbf{T}(\mathbf{v_1} + \mathbf{v_2}) = \mathbf{T}(\mathbf{v_1}) + \mathbf{T}(\mathbf{v_2})$.
2. $\mathbf{T}(\alpha\mathbf{v}) = \alpha\mathbf{T}(\mathbf{v})$ for any $\alpha \in \mathbb{R}, \mathbf{v} \in \mathcal{V}$.

Range of $\mathbf{T} = \mathbf{T}(\mathcal{V})$, kernel of $\mathbf{T} = \{\mathbf{v} \mid \mathbf{T}(\mathbf{v}) = 0\}$. If $\mathbf{w_1}, \mathbf{w_2} \in \mathbf{T}(\mathcal{V})$, then $\mathbf{w_1} = \mathbf{T}(\mathbf{v_1}), \mathbf{w_2} = \mathbf{T}(\mathbf{v_2})$ for some $\mathbf{v_1}, \mathbf{v_2} \in \mathcal{V}$, $\alpha\mathbf{w_1} + \beta\mathbf{w_2} = \alpha\mathbf{T}(\mathbf{v_1}) + \beta\mathbf{T}(\mathbf{v_2}) = \mathbf{T}(\alpha\mathbf{v_1} + \beta\mathbf{v_2})$. But $\alpha\mathbf{v_1} + \beta\mathbf{v_2} \in \mathcal{V} \therefore \alpha\mathbf{w_1} + \beta\mathbf{w_2} \in \mathbf{T}(\mathcal{V})$, thus $\mathbf{T}(\mathcal{V})$ is a subspace of $\mathcal{W}$. Note that $\mathbf{T}(\mathcal{V}) \neq \emptyset \because \mathbf{0} \in \mathbf{T}(\mathcal{V})$ so $\mathbf{T}(\mathcal{V})$ is a vector space.

If $\mathbf{v_1}, \mathbf{v_2} \in \operatorname{kernel} \mathbf{T}$ then $\mathbf{T}(\mathbf{v_1}) = \mathbf{0}, \mathbf{T}(\mathbf{v_2}) = \mathbf{0}$. Now $\mathbf{T}(\alpha\mathbf{v_1} + \beta\mathbf{v_2}) = \alpha\mathbf{T}(\mathbf{v_1}) + \beta\mathbf{T}(\mathbf{v_2}) = \mathbf{0} \Rightarrow \alpha\mathbf{v_1} + \beta\mathbf{v_2} \in \operatorname{kernel}\mathbf{T}$. Thus $\operatorname{kernel}\mathbf{T}$ is a subspace. $\operatorname{kernel}\mathbf{T} \neq \emptyset, bf0 \in \operatorname{kernel}\mathbf{T}$ so $\operatorname{kernel}\mathbf{T}$ is a vector space. ∎

**Question 2(a)** *Reduce the matrix*

$$\begin{pmatrix} 2 & 3 & -1 & 0 \\ 1 & -1 & 2 & 0 \\ 1 & 2 & -1 & 0 \end{pmatrix}$$

*to row echelon form.*

**Solution.** Let the given matrix be called $\mathbf{A}$.

Operation $R_1 - R_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 4 & -3 & 0 \\ 1 & -1 & 2 & 0 \\ 1 & 2 & -1 & 0 \end{pmatrix}$

Operation $R_2 - R_1, R_3 - R_1 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 4 & -3 & 0 \\ 0 & -5 & 5 & 0 \\ 0 & -2 & 2 & 0 \end{pmatrix}$

Operation $-\frac{1}{5}R_2, -\frac{1}{2}R_3 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 4 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & -1 & 0 \end{pmatrix}$

Operation $R_3 - R_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 4 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Operation $R_1 - 4R_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

Thus $\operatorname{rank}\mathbf{A} = 2$ and the row echelon form is $\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ ∎

**Question 2(b)** *If $\mathcal{V}$ is a vector space of dimension $n$ and $\mathbf{T}$ is a linear transformation on $\mathcal{V}$ of rank $r$, prove that $\mathbf{T}$ has nullity $n - r$.*

**Solution.** See 1998 question 3(a). ∎

**Question 2(c)** *Show that the system of equations*

$$\begin{aligned} 3x + y - 5z &= -1 \\ x - 2y + z &= -5 \\ x + 5y - 7z &= 2 \end{aligned}$$

*is inconsistent.*

**Solution.** From the first two equations, $(3x + y - 5z) - 2(x - 2y + z) = -1 - 2(-5) = 9 \Rightarrow x + 5y - 7z = 9$. But this is inconsistent with the third equation, hence the overall system in inconsistent. ∎

**Question 3(a)** *Prove that the trace of a matrix is equal to the sum of its characteristic roots.*

**Solution.** The characteristic polynomial of $\mathbf{A}$ is $|\lambda\mathbf{I} - \mathbf{A}| = \lambda^n + p_1\lambda^{n-1} + p_2\lambda^{n-2} + \ldots + p_n$. Thus the sum of the roots of $|\lambda\mathbf{I} - \mathbf{A}| = -p_1 = a_{11} + a_{22} + \ldots + a_{nn} = \operatorname{tr} \mathbf{A}$. Thus the trace of $\mathbf{A}$ = sum of the eigenvalues of $\mathbf{A}$. ∎

**Question 3(b)** *If* $\mathbf{A}, \mathbf{B}$ *are two non-singular matrices of the same order, prove that* $\mathbf{AB}$ *and* $\mathbf{BA}$ *have the same eigenvalues.*

**Solution.** See 1995 question 2(b). ∎

**Question 3(c)** *Find the eigenvalues and eigenvectors of the matrix* $\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

**Solution.** The characteristic equation of $\mathbf{A}$ is $(\cos\theta - \lambda)(-\cos\theta - \lambda) - \sin^2\theta = 0 \Rightarrow \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1$.

If $(x_1, x_2)$ is an eigenvector for $\lambda = 1$, then

$$\begin{pmatrix} \cos\theta - 1 & \sin\theta \\ \sin\theta & -\cos\theta - 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

Thus $x_1(\cos\theta - 1) + x_2 \sin\theta = 0, x_1 \sin\theta + x_2(-\cos\theta - 1) = 0$. We can take $x_1 = 1 + \cos\theta, x_2 = \sin\theta$.

Similarly if $(x_1, x_2)$ is an eigenvector for $\lambda = -1$, then

$$\begin{pmatrix} \cos\theta + 1 & \sin\theta \\ \sin\theta & -\cos\theta + 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0}$$

Thus $x_1(\cos\theta + 1) + x_2 \sin\theta = 0, x_1 \sin\theta + x_2(-\cos\theta + 1) = 0$. We can take $x_1 = 1 - \cos\theta, x_2 = -\sin\theta$ as an eigenvector. ∎

**Question 4(a)** *If $\mathcal{V}$ is finite dimensional and if $\mathcal{W}$ is a subspace of $\mathcal{V}$, then show that $\mathcal{W}$ is finite dimensional and $\dim \mathcal{W} \leq \dim \mathcal{V}$.*

**Solution.** If $\mathcal{W} = \{\mathbf{0}\}$ then $\dim \mathcal{W} = 0 \leq \dim \mathcal{V}$. If $\mathcal{W} \neq \{\mathbf{0}\}$, let $\mathbf{v_1} \in \mathcal{W}, \mathbf{v_1} \neq \mathbf{0}$. Let $\mathcal{W}_1$ be the space spanned by $\mathbf{v_1}$ then $\mathcal{W}_1$ is of dimension 1. If $\mathcal{W}_1 = \mathcal{W}$, then $\dim \mathcal{W} = 1 \leq \dim \mathcal{V}$.

If $\mathcal{W}_1 \neq \mathcal{W}$, then there exists a $\mathbf{v_2} \in \mathcal{W}, \mathbf{v_2} \notin \mathcal{W}_1$. $\mathbf{v_1}, \mathbf{v_2}$ are linearly independent — if $a\mathbf{v_1} + b\mathbf{v_2} = 0$, then if $b \neq 0$ then $\mathbf{v_2} = -\frac{a}{b}\mathbf{v_1} \Rightarrow \mathbf{v_2} \in \mathcal{W}_1$, which is not true, hence $b = 0 \Rightarrow a = 0$. Now let $\mathcal{W}_2$ be the space spanned by $\mathbf{v_1}, \mathbf{v_2}$ then $\mathcal{W}_2$ is of dimension 2. If $\mathcal{W}_2 = \mathcal{W}$, then $\dim \mathcal{W} = 2 \leq \dim \mathcal{V}$.

We continue the same reasoning as above, but this process must stop after at most $r$ steps where $r \leq n$, otherwise we would have found $n+1$ linearly independent vectors in $\mathcal{V}$, which is not possible. After $r$ steps, we would have $\mathbf{v_1}, \ldots, \mathbf{v_r}$ which are linearly independent and span $\mathcal{W}$. Thus $\dim \mathcal{W} \leq \dim \mathcal{V}$, and $\mathcal{W}$ is finite dimensional. ∎

**Question 5(a)** *State and prove the Cayley-Hamilton Theorem when the eigenvalues are all different.*

**Solution.** See 1987 question 5(a). ∎

**Question 5(b)** *When are two real symmetric matrices said to be congruent? Is congruence an equivalence relation? Also show how you can find the signature of $\mathbf{A}$.*

**Solution.** Two matrices $\mathbf{A}, \mathbf{B}$ are said to be congruent to each other if there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{B}$.

Congruence is an equivalence relation:

- Reflexive: $\mathbf{A} \equiv \mathbf{A} \because \mathbf{A} = \mathbf{I}'\mathbf{A}\mathbf{I}$, $\mathbf{I}$ is the unit matrix.
- Symmetric: $\mathbf{A} \equiv \mathbf{B} \Rightarrow \mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{B} \Rightarrow \mathbf{A} = (\mathbf{P}^{-1})'\mathbf{B}\mathbf{P}^{-1} \Rightarrow \mathbf{B} \equiv \mathbf{A}$.
- Transitive: $\mathbf{A} \equiv \mathbf{B}, \mathbf{B} \equiv \mathbf{C} \Rightarrow \mathbf{A} \equiv \mathbf{C}$ — $\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{B}, \mathbf{Q}'\mathbf{B}\mathbf{Q} = \mathbf{C} \Rightarrow \mathbf{Q}'\mathbf{P}'\mathbf{A}\mathbf{P}\mathbf{Q} = \mathbf{C} \Rightarrow \mathbf{A} \equiv \mathbf{C}$ because $\mathbf{P}\mathbf{Q}$ is nonsingular as both $\mathbf{P}, \mathbf{Q}$ are nonsingular.

Given a symmetric matrix $\mathbf{A}$, we first prove that there exists a nonsingular matrix $\mathbf{P}$ such that $\mathbf{P}'\mathbf{A}\mathbf{P} = \text{diagonal}[\alpha_1, \alpha_2, \ldots, \alpha_r, 0, \ldots, 0]$ where $r$ is the rank of $\mathbf{A}$.

We will prove this by induction on the order $n$ of the matrix $\mathbf{A}$. If $n = 1$, there is nothing to prove. Assume that the result is true for all matrices of order $< n$.

Step 1. We first ensure that we have $a_{11} \neq 0$. If it is 0, but some other $a_{kk} \neq 0$, we interchange the $k$-th row with the first row and the $k$-th column with the first column, to get $\mathbf{B} = \mathbf{P}'\mathbf{A}\mathbf{P}$, where $b_{11} = a_{kk} \neq 0$. Note that $\mathbf{P}$ is the elementary matrix $\mathbf{E}_{1k}$ (see 1983 question 2(a)), and is hence nonsingular and symmetric, so $\mathbf{B}$ is symmetric.

If all $a_{ii}$ are 0, but some $a_{ij} \neq 0$. We add the $j$-th row to the $i$-th row and the $j$-the column to the $i$-th column by multiplying $\mathbf{A}$ by $\mathbf{E}_{ij}(1)$ and its transpose, to get $\mathbf{B} = \mathbf{E}_{ij}(1)\mathbf{A}\mathbf{E}_{ij}(1)'$

— now $b_{ii} = a_{ij} + a_{ji} \neq 0$. $B$ is still symmetric, and we can shift $b_{ii}$ to the leading place as above.

(Note that if all $a_{ij} = 0$, we stop.)

Thus we start with $a_{11} \neq 0$. We subtract $\frac{a_{1k}}{a_{11}}$ times the first row from the $k$-th row and $\frac{a_{1k}}{a_{11}}$ times the first column from the $k$-th column, by performing $\mathbf{B} = \mathbf{E}_{k1}(-\frac{a_{1k}}{a_{11}})\mathbf{A}\mathbf{E}_{k1}(-\frac{a_{1k}}{a_{11}})'$ Repeating this for all $k$, $2 \leq k \leq n$, we get $\mathbf{P_1'AP_1} = \begin{pmatrix} a_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{A}_1 \end{pmatrix}$, where $\mathbf{A}_1$ is $n-1 \times n-1$ and $\mathbf{P_1}$ is nonsingular. Now by induction, $\exists \mathbf{P_2}, n-1 \times n-1$ such that $\mathbf{P_2'AP_2} =$ diagonal$[\beta_2, \ldots, \beta_r, 0, \ldots, 0]$, $\operatorname{rank} \mathbf{A_1} = \operatorname{rank} \mathbf{A} - 1$. Now set $\mathbf{P} = \mathbf{P}_1 \begin{pmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{P}_2 \end{pmatrix}$ to get the result.

Now that we have $\mathbf{P'AP} =$ diagonal$[\alpha_1, \alpha_2, \ldots, \alpha_r, 0, \ldots, 0]$, let us assume that $\alpha_1, \ldots, \alpha_s$ are positive, the rest are negative. Then let $\alpha_i = \beta_i^2, 1 \leq i \leq s, -\alpha_j = \beta_j^2, s+1 \leq j \leq r$. Set $\mathbf{Q} =$ diagonal$[\beta_1^{-1}, \ldots, \beta_r^{-1}, 1, \ldots, 1]$. Then $\mathbf{x'Q'P'APQx} = x_1^2 + \ldots + x_s^2 - x_{s+1}^2 - x_r^2$. Thus we can find the signature of $\mathbf{A}$ by looking at the number of positive and negative squares of the RHS. ∎

**Question 5(c)** *Derive a set of necessary and sufficient conditions that the real quadratic form $\sum_{j=1}^3 \sum_{i=1}^3 a_{ij}x_ix_j$ be positive definite.*

*Is $4x^2 + 9y^2 + 2z^2 + 8yz + 6zx + 6xy$ positive definite?*

**Solution.** For the first part, see 1992 question 2(c).

$$\begin{aligned}
Q(x,y,z) &= 4x^2 + 9y^2 + 2z^2 + 8yz + 6zx + 6xy \\
&= (2x + \frac{3}{2}y + \frac{3}{2}z)^2 + 9y^2 + 2z^2 + 8yz + -\frac{9}{2}yz - \frac{9}{4}y^2 - \frac{9}{4}z^2 \\
&= (2x + \frac{3}{2}y + \frac{3}{2}z)^2 + \frac{27}{4}y^2 - \frac{1}{4}z^2 - \frac{7}{2}yz \\
&= (2x + \frac{3}{2}y + \frac{3}{2}z)^2 + \frac{27}{4}(y^2 - \frac{1}{27}z^2 - \frac{14}{27}yz) \\
&= (2x + \frac{3}{2}y + \frac{3}{2}z)^2 + \frac{27}{4}(y - \frac{7}{27}z)^2 - \frac{1}{4}z^2 - \frac{49}{108}z^2
\end{aligned}$$

So set $X = 2x + \frac{3}{2}y + \frac{3}{2}z, Y = y - \frac{7}{27}z, Z = z$, then $Q(x,y,z)$ is transformed to $X^2 + \frac{27}{4}Y^2 - \frac{76}{108}Z^2$. Hence $Q(x,y,z)$ is is not positive definite. ∎