UPSC Civil Services Main 1984 - Mathematics
Linear Algebra

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Question 1(a) *If $\mathcal{W}_1, \mathcal{W}_2$ are finite dimensional subspaces of a vector space $\mathcal{V}$, then show that $\mathcal{W}_1 + \mathcal{W}_2$ is finite dimensional and* $\dim \mathcal{W}_1 + \dim \mathcal{W}_2 = \dim(\mathcal{W}_1 + \mathcal{W}_2) + \dim(\mathcal{W}_1 \cap \mathcal{W}_2)$.

Solution. See 1988, question 1(b). ∎

Question 1(b) *If $\mathbf{A}$ and $\mathbf{B}$ are n-rowed square non-zero matrices such that $\mathbf{AB} = \mathbf{0}$, then show that both $\mathbf{A}$ and $\mathbf{B}$ are singular. If both $\mathbf{A}$ and $\mathbf{B}$ are singular, and $\mathbf{AB} = \mathbf{0}$, does it imply that $\mathbf{BA} = \mathbf{0}$? Justify your answer.*

Solution. If $\mathbf{A}$ were non-singular, then $\mathbf{A}^{-1}\mathbf{AB} = \mathbf{0} \Rightarrow \mathbf{B} = \mathbf{0}$. Thus $\mathbf{A}$ is singular, and similarly $\mathbf{B}$ is singular.

$\mathbf{AB} = \mathbf{0}$ does not imply that $\mathbf{BA} = \mathbf{0}$. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Then $\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$, but $\mathbf{BA} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \neq \mathbf{0}$. ∎

Question 2(a) *Show that row-equivalent matrices have the same rank.*

Solution. See 1986 question 3(b). ∎

Question 2(b) *A linear transformation T on a vector space $\mathcal{V}$ with finite basis $\alpha_1, \alpha_2, \dots, \alpha_n$ is non-singular if and only if the vectors $\alpha_1 T, \alpha_2 T, \dots, \alpha_n T$ are linearly independent in $\mathcal{V}$. When this is the case, show that T has an inverse T^{-1} with $TT^{-1} = T^{-1}T = I$.*

Solution. If $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$ are linearly independent, then T is one-one: Let $\mathbf{v} \in \mathcal{V}$. Then $\mathbf{v} = \sum_{i=1}^n a_i\alpha_i, T(\mathbf{v}) = \sum_{i=1}^n a_i T(\alpha_i)$. If $T(\mathbf{v}) = 0$, then $\sum_{i=1}^n a_i T(\alpha_i) = 0 \Rightarrow a_i = 0, 1 \le i \le n$, because $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$ are linearly independent. Thus $T(\mathbf{v}) = 0 \Rightarrow \mathbf{v} = \mathbf{0}$, so T is one-one.

T is onto: $\dim T(\mathbf{v}) = \dim \mathcal{V} = \mathbf{n}$.

Thus T is invertible, in fact $T^{-1}(T(\alpha_i)) = \alpha_i$.

T^{-1} is a linear transformation: Let $T^{-1}(\mathbf{v}) = \mathbf{u}, \mathbf{T^{-1}}(\mathbf{w}) = \mathbf{x}$. Then $T(\mathbf{u}) = \mathbf{v}, T(\mathbf{x}) = \mathbf{w}$. Let $T^{-1}(a\mathbf{v} + b\mathbf{w}) = \mathbf{z}$, then $T(\mathbf{z}) = a\mathbf{v} + b\mathbf{w} = aT(\mathbf{u}) + bT(\mathbf{x}) = T(a\mathbf{u} + b\mathbf{x}) \Rightarrow \mathbf{z} = a\mathbf{u} + b\mathbf{x}$. Thus $T^{-1}(a\mathbf{v} + b\mathbf{w}) = aT^{-1}(\mathbf{v}) + bT^{-1}(\mathbf{w})$, so T^{-1} is linear. It is obvious that $TT^{-1} = T^{-1}T = I$, as this is true for the basis elements by definition, and extends to all vectors by linearity.

Conversely if T is non-singular, then $a_1T(\alpha_1) + a_2T(\alpha_2) + \ldots + a_nT(\alpha_n) = 0 \Rightarrow T(\sum_{i=1}^n a_i\alpha_i) = 0 \Rightarrow \sum_{i=1}^n a_i\alpha_i = 0 \Rightarrow a_i = 0, 1 \le i \le n$ because $\alpha_1, \alpha_2, \ldots, \alpha_n$ are linearly independent. Thus $T(\alpha_1), T(\alpha_2), \ldots, T(\alpha_n)$ are linearly independent. ∎

Question 2(c) *Solve the following system of equations:*

$$\begin{aligned} 3x_1 + 2x_2 + 2x_3 - 5x_4 &= 8 \\ 2x_1 + 5x_2 + 5x_3 - 18x_4 &= 9 \\ 4x_1 - x_2 - x_3 + 8x_4 &= 7 \end{aligned}$$

Solution. Let the coefficient matrix be

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & 2 & -5 \\ 2 & 5 & 5 & -18 \\ 4 & -1 & -1 & 8 \end{pmatrix}$$

Doubling R_1, and subtracting $R_2 + R_3$, we get

$$\mathbf{A} \sim \begin{pmatrix} 0 & 0 & 0 & 0 \\ 2 & 5 & 5 & -18 \\ 4 & -1 & -1 & 8 \end{pmatrix}$$

Thus the rank of $\mathbf{A}$ is 2.

The augmented matrix

$$\mathbf{B} = \begin{pmatrix} 3 & 2 & 2 & -5 & 8 \\ 2 & 5 & 5 & -18 & 9 \\ 4 & -1 & -1 & 8 & 7 \end{pmatrix} \sim \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 2 & 5 & 5 & -18 & 9 \\ 4 & -1 & -1 & 8 & 7 \end{pmatrix}$$

Thus the rank of $\mathbf{B}$ is also 2, so the system is consistent.

Since the rank of $\mathbf{A}$ is 2, the space of solutions has rank $= 4 - 2 = 2$. Adding twice the third equation to the first we get $11x_1 + 11x_4 = 22 \Rightarrow x_1 = 2 - x_4$. Substituting this in the third equation, we get $x_2 = 1 - x_3 + 4x_4$. Thus the required solution system is $(2 - x_4, 1 - x_3 + 4x_4, x_3, x_4)$, where x_3, x_4 take any value in $\mathbb{R}$. ∎

Question 3(a) *Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over the field F, and let T be a linear transformation from $\mathcal{V}$ to $\mathcal{W}$. If $\mathcal{V}$ is finite dimensional show that* $\operatorname{rank} T + \operatorname{nullity} T = \dim \mathcal{V}$.

Solution. See question 1(a) from 1992. ∎

Question 3(b) *Let $\mathbf{A}$ be a square matrix and $\mathbf{T}$ be non-singular. Let $\widetilde{\mathbf{A}} = \mathbf{T}^{-1}\mathbf{A}\mathbf{T}$. Show that*

1. *$\mathbf{A}$ and $\widetilde{\mathbf{A}}$ have the same eigenvalues.*
2. $\operatorname{tr} \mathbf{A} = \operatorname{tr} \widetilde{\mathbf{A}}$.
3. *If $\mathbf{x}$ is an eigenvector of $\mathbf{A}$ corresponding to an eigenvalue, then $\mathbf{T}^{-1}\mathbf{x}$ is an eigenvector of $\widetilde{\mathbf{A}}$ corresponding to the same eigenvalue.*

Solution.

1. The eigenvalues of $\mathbf{A}$ are roots of $|x\mathbf{I} - \mathbf{A}| = 0$. The eigenvalue of $\widetilde{\mathbf{A}}$ are roots of $0 = |x\mathbf{I} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T}| = |\mathbf{T}^{-1}x\mathbf{I}\mathbf{T} - \mathbf{T}^{-1}\mathbf{A}\mathbf{T}| = |\mathbf{T}^{-1}||x\mathbf{I} - \mathbf{A}||\mathbf{T}| = |x\mathbf{I} - \mathbf{A}|$, so the eigenvalues are the same.
2. $\operatorname{tr} \mathbf{AB} = \operatorname{tr} \mathbf{BA}$, so $\operatorname{tr} \mathbf{T}^{-1}\mathbf{A}\mathbf{T} = \operatorname{tr} \mathbf{A}\mathbf{T}\mathbf{T}^{-1} = \operatorname{tr} \mathbf{A}$.
3. If $\mathbf{A}\mathbf{x} = \lambda\mathbf{x}$ then $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}(\mathbf{T}^{-1}\mathbf{x}) = \mathbf{T}^{-1}(\lambda\mathbf{x}) = \lambda\mathbf{T}^{-1}\mathbf{x}$.

∎

Question 3(c) *A 3×3 matrix has the eigenvalues 6, 2, -1. The corresponding eigenvectors are $(2, 3, -2), (9, 5, 4), (4, 4, -1)$. Find the matrix.*

Solution. Let $\mathbf{P} = \begin{pmatrix} 2 & 9 & 4 \\ 3 & 5 & 4 \\ -2 & 4 & -1 \end{pmatrix}$, and let $\mathbf{A}$ be the required matrix. Then $\mathbf{AP} = \mathbf{P}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$, therefore $\mathbf{A} = \mathbf{P}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}\mathbf{P}^{-1}$. A simple calculation gives $\mathbf{P}^{-1} = \begin{pmatrix} -21 & 25 & 16 \\ -5 & 6 & 4 \\ 22 & -26 & -17 \end{pmatrix}$, note that $|\mathbf{P}| = 1$. Now $\begin{pmatrix} 2 & 9 & 4 \\ 3 & 5 & 4 \\ -2 & 4 & -1 \end{pmatrix}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 12 & 18 & -4 \\ 18 & 10 & -4 \\ -12 & 8 & 1 \end{pmatrix}$.
Thus $\mathbf{A} = \begin{pmatrix} 12 & 18 & -4 \\ 18 & 10 & -4 \\ -12 & 8 & 1 \end{pmatrix}\begin{pmatrix} -21 & 25 & 16 \\ -5 & 6 & 4 \\ 22 & -26 & -17 \end{pmatrix} = \begin{pmatrix} -430 & 512 & 352 \\ 516 & 620 & 396 \\ 234 & 226 & -173 \end{pmatrix}$

A longer way would be to set $\mathbf{A} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix}$. Then $\mathbf{AP} = \mathbf{P}\begin{pmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. This yields three systems of linear equations which have to be solved. ∎

Question 4(a) *Let $\mathcal{V}$ be the set of all functions from a non-empty set into a field K. For any function $f, g \in \mathcal{V}$ and any scalar $k \in K$, let $f + g$ and kf be functions in $\mathcal{V}$ defined by $(f + g)(x) = f(x) + g(x), (kf)(x) = kf(x)$ for every $x \in X$. Prove that $\mathcal{V}$ is a vector space over K.*

Solution. $\mathcal{V} = \{f \mid f : X \longrightarrow K\}$.

1. $f, g \in \mathcal{V} \Rightarrow (f + g)(x) = f(x) + g(x) \Rightarrow f + g : X \longrightarrow K \Rightarrow f + g \in \mathcal{V}$.
2. The zero function namely $0(x) = 0 \forall x \in X$ is the additive identity of $\mathcal{V}$ i.e. $f + 0 = 0 + f = f \forall f \in \mathcal{V}$.
3. $f \in \mathcal{V} \Rightarrow -f \in \mathcal{V}$ where $(-f)(x) = -f(x)$ anf $f + (-f) = 0 = (-f) + f$.
4. $(f + g) + h = f + (g + h)$ for every $f, g, h \in \mathcal{V}$.
5. If $f \in \mathcal{V}, k \in K$, then $(kf)(x) = kf(x) \forall x \in X$, so $kf \in \mathcal{V}$ and $k(f + g) = kf + kg$.
6. If $k, k' \in K, f \in \mathcal{V}$, then $k(k'f) = (kk')f$.
7. If $1 \in K$ is the multiplicative identity, then $1f = f$ for every f.
8. $k, k' \in K, f \in \mathcal{V} \Rightarrow (k + k')f(x) = kf(x) + k'f(x) \Rightarrow (k + k')f = kf + k'f$.

Thus $\mathcal{V}$ is a vector space over K. ∎

Question 4(b) *Find the eigenvalues and basis for each eigenspace of the matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & -3 & 3 \\ 3 & -5 & 3 \\ 6 & -6 & 4 \end{pmatrix}$$

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Solution. The characteristic equation of $\mathbf{A}$ is

$$\begin{array}{rcrcl} & & |\mathbf{A} - \lambda\mathbf{I}| & = & 0 \\ \Rightarrow & & \begin{pmatrix} 1-\lambda & -3 & 3 \\ 3 & -5-\lambda & 3 \\ 6 & -6 & 4-\lambda \end{pmatrix} & = & 0 \\ \Rightarrow & & (1-\lambda)(\lambda+5)(\lambda-4) + 18(1-\lambda) + 3(12 - 3\lambda - 18) + 3(-18 + 30 + 6\lambda) & = & 0 \\ \Rightarrow & & (1-\lambda)(\lambda^2 + \lambda - 20) + 18 - 18\lambda - 9\lambda - 18 + 36 + 18\lambda & = & 0 \\ \Rightarrow & & \lambda^2 + \lambda - 20 - \lambda^3 - \lambda^2 + 20\lambda - 9\lambda + 36 & = & 0 \\ \Rightarrow & & \lambda^3 - 12\lambda - 16 & = & 0 \end{array}$$

Thus $\lambda = -2, 4, -2$. Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 4$.

$$\begin{pmatrix} -3 & -3 & 3 \\ 3 & -9 & 3 \\ 6 & -6 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Thus $-3x_1 - 3x_2 + 3x_3 = 0, 3x_1 - 9x_2 + 3x_3 = 0, 6x_1 - 6x_2 = 0 \Rightarrow x_1 = x_2, x_3 = 2x_1$. We can take $(1, 1, 2)$ as an eigenvector corresponding to $\lambda = 4$.

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = -2$.

$$\begin{pmatrix} 3 & -3 & 3 \\ 3 & -3 & 3 \\ 6 & -6 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Thus $3x_1 - 3x_2 + 3x_3 = 0 \Rightarrow x_2 = x_1 + x_3$. $(1, 1, 0), (0, 1, 1)$ can be taken as eigenvectors for $\lambda = -2$.

Clearly $(1, 1, 2)$ is a basis for the eigenspace for $\lambda = 4$. $(1, 1, 0), (0, 1, 1)$ is a basis for the eigenspace for $\lambda = -2$. ∎

Question 4(c) *Let a vector space $\mathcal{V}$ have finite dimension and let $\mathcal{W}$ be a subspace of $\mathcal{V}$ and $\mathcal{W}^0$ the annihilator of $\mathcal{W}$. Prove that* $\dim \mathcal{W} + \dim \mathcal{W}^0 = \dim \mathcal{V}$.

Solution. Let $\dim \mathcal{V} = n$, $\dim \mathcal{W} = m$, $\mathcal{W} \subseteq \mathcal{V}$. Let $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ be a basis of $\mathcal{V}$ so chosen that $\{\mathbf{v}_1, \ldots, \mathbf{v}_m\}$ is a basis of $\mathcal{W}$. Let $\{v_1^*, \ldots, v_n^*\}$ be the dual basis of $\mathcal{V}^*$ i.e. $v_i^*(\mathbf{v}_j) = \delta_{ij}$. We shall show that $\mathcal{W}^0$ has $\{v_{m+1}^*, \ldots, v_n^*\}$ as a basis.

By definition of the dual basis $v_i^*(\mathbf{v}_j) = 0$ when $1 \le i \le m$ and $m + 1 \le j \le n$. Since $v_j^*, m + 1 \le j \le n$ annihilate the basis of $\mathcal{W}$, it follows that $v_j(\mathbf{w}) = \mathbf{0}$ for all $\mathbf{w} \in \mathcal{W}$. Thus $\{v_{m+1}^*, \ldots, v_n^*\} \subseteq \mathcal{W}^0$, and are linearly independent, being a subset of a linearly independent set.

Let $f \in \mathcal{W}^0$, then $f = \sum_{i=1}^n a_i v_i^*$. We shall show that $a_i = 0$ for $1 \le i \le m$, thus f is a linear combination of $\{v_{m+1}^*, \ldots, v_n^*\}$. By definition of $\mathcal{W}^0$, $f(\mathbf{v}_1) = 0, \ldots, f(\mathbf{v}_m) = 0$, therefore $(\sum_{i=1}^n a_i v_i^*)(\mathbf{v}_j) = \sum_{i=1}^n a_i \delta_{ij} = a_j = 0$ when $1 \le j \le m$. Thus $\{v_{m+1}^*, \ldots, v_n^*\}$ is a basis of $\mathcal{W}^0$, hence $\dim \mathcal{W}^0 = n - m$, hence $\dim \mathcal{W} + \dim \mathcal{W}^0 = n = \dim \mathcal{V}$. ∎

Question 5(a) *Prove that every matrix satisfies its characteristic equation.*

Solution. See 1987 question 3(a). ∎

Question 5(b) *Find a necessary and sufficient condition that the real quadratic form* $\displaystyle\sum_{i=1}^n \sum_{j=1}^n a_{ij} x_i x_j$ *be a positive definite form.*

Solution. See 1991 question 1(c) and 1992 question 1(c) ∎

Question 5(c) *Prove that the rank of the product of two matrices cannot exceed the rank of either of them.*

Solution. See 1987 question 1(b). ∎