UPSC Civil Services Main 1985 - Mathematics Linear Algebra

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Question 1(a) *If $\mathcal{W}_1$ and $\mathcal{W}_2$ are finite dimensional subspaces of a vector space $\mathcal{V}$, then show that $\mathcal{W}_1 + \mathcal{W}_2$ is finite dimensional and*

$$\dim \mathcal{W}_1 + \dim \mathcal{W}_2 = \dim(\mathcal{W}_1 + \mathcal{W}_2) + \dim(\mathcal{W}_1 \cap \mathcal{W}_2)$$

Solution. See 1988 question 1(b). ∎

Question 1(b) *Let $M_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, M_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, M_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, M_4 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Prove that the set $\{M_1, M_2, M_3, M_4\}$ forms the basis of the vector space of 2×2 matrices.*

Solution. See 2006 question 1(a). ∎

Question 1(c) *Find the inverse of the matrix $\mathbf{A} = \begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix}$.*

Solution.

$$\begin{pmatrix} 1 & 3 & 3 \\ 1 & 4 & 3 \\ 1 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{A}$$

Using operation $R_2 - R_1, R_3 - R_1$, we get

$$\begin{pmatrix} 1 & 3 & 3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \mathbf{A}$$

Now with operation $R_1 - 3(R_2 + R_3)$ we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} \mathbf{A}$$

Thus the inverse of $\mathbf{A}$ is $\begin{pmatrix} 7 & -3 & -3 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$. ∎

Question 2(a) *If $T : \mathcal{V} \longrightarrow \mathcal{W}$ is a linear transformation from an n-dimensional vector space $\mathcal{V}$ to a vector space $\mathcal{W}$, then prove that rank(T) + nullity(T) = n.*

Solution. See 1992 question 1(a). ∎

Question 2(b) *Consider the basis $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ of $\mathbb{R}^3$, where $\mathbf{v}_1 = (1,1,1), \mathbf{v}_2 = (1,1,0), \mathbf{v}_3 = (1,0,0)$. Express $(2,-3,5)$ in terms of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Let $T : \mathbb{R}^3 \longrightarrow \mathbb{R}^2$ be defined as $T(\mathbf{v}_1) = (1,0), T(\mathbf{v}_2) = (2,-1), T(\mathbf{v}_3) = (4,3)$. Find $T(2,-3,5)$.*

Solution. Let $(2,-3,5) = a(1,1,1) + b(1,1,0) + c(1,0,0)$. Then $a + b + c = 2, a + b = -3, a = 5 \Rightarrow a = 5, b = -8, c = 5$. Thus $(2,-3,5) = 5\mathbf{v}_1 - 8\mathbf{v}_2 + 5\mathbf{v}_3$.

$T(2,-3,5) = 5T(\mathbf{v}_1) - 8T(\mathbf{v}_2) + 5T(\mathbf{v}_3) = 5(1,0) - 8(2,-1) + 5(4,3) = (9,23)$. ∎

Question 2(c) *Reduce the following matrix into echelon form:* $\begin{pmatrix} 6 & 3 & -4 \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{pmatrix}$.

Solution. $\mathbf{A} = \begin{pmatrix} 6 & 3 & -4 \\ -4 & 1 & -6 \\ 1 & 2 & -5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & -5 \\ -4 & 1 & -6 \\ 6 & 3 & -4 \end{pmatrix}$ by exchanging R_1 and R_3.

Now $R_2 + 4R_1, R_3 - 6R_1 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & -9 & 26 \end{pmatrix}$.

$R_3 + R_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 2 & -5 \\ 0 & 9 & -26 \\ 0 & 0 & 0 \end{pmatrix}$.

Multiply R_2 by $\frac{1}{9}$ to get $\mathbf{A} \sim \begin{pmatrix} 1 & 2 & -5 \\ 0 & 1 & -\frac{26}{9} \\ 0 & 0 & 0 \end{pmatrix}$.

Now $R_1 - 2R_2 \Rightarrow \mathbf{A} \sim \begin{pmatrix} 1 & 0 & \frac{7}{9} \\ 0 & 1 & -\frac{26}{9} \\ 0 & 0 & 0 \end{pmatrix}$ which is the required form. ∎

Question 3(a) *Show that if λ is an eigenvalue of matrix $\mathbf{A}$, then λ^n is an eigenvalue of $\mathbf{A}^n$, where n is a positive integer.*

Solution. If $\mathbf{x}$ is an eigenvector for λ, then $\mathbf{A}^n\mathbf{x} = \mathbf{A}^{n-1}\mathbf{A}\mathbf{x} = \lambda\mathbf{A}^{n-1}\mathbf{x}$. Repeating this process, we get the result. ∎

Question 3(b) *Determine if the vectors $(1, -2, 1), (2, 1, -1), (7, -4, 1)$ are linearly independent in $\mathbb{R}^3$.*

Solution. If possible, let $a(1, -2, 1) + b(2, 1, -1) + c(7, -4, 1) = \mathbf{0}$. Then $a + 2b + 7c = 0, -2a + b - 4c = 0, a - b + c = 0$. Adding the last two we get $a = -3c$, and from the third we then get $b = -2c$. These values satisfy the first equation also, hence letting $c = -1$ we get $3(1, -2, 1) + 2(2, 1, -1) - (7, -4, 1) = \mathbf{0}$. Thus the vectors are linearly dependent. ∎

Question 3(c) *Solve*

$$
\begin{aligned}
2x_1 + 3x_2 + x_3 &= 9 && (1)\\
x_1 + 2x_2 + 3x_3 &= 6 && (2)\\
3x_1 + x_2 + 2x_3 &= 8 && (3)
\end{aligned}
$$

Solution. $2(2) - (1) \Rightarrow x_2 + 5x_3 = 3 \Rightarrow x_2 = 3 - 5x_3$. Substituting x_2 in (2), $x_1 = 7x_3$. Now substituting x_1, x_2 in (3), we get $21x_3 + 3 - 5x_3 + 2x_3 = 8 \Rightarrow x_3 = \frac{5}{18}, x_2 = \frac{29}{18}, x_1 = \frac{35}{18}$, which is the required solution.

(Using Cramer's rule would have been lengthy.) ∎

Paper II

Question 4(a) *Let $\mathcal{V}$ be the vector space of all functions from $\mathbb{R}$ into $\mathbb{R}$. Let $\mathcal{V}_e$ be the subset of all even functions $f, f(-x) = f(x)$, and $\mathcal{V}_o$ be the subset of all odd functions $f, f(-x) = -f(x)$. Prove that*

1. *$\mathcal{V}_e$ and $\mathcal{V}_o$ are subspaces of $\mathcal{V}$*
2. *$\mathcal{V}_e + \mathcal{V}_o = \mathcal{V}$*
3. *$\mathcal{V}_e \cap \mathcal{V}_o = \{0\}$*

Solution.

1. Let $f, g \in \mathcal{V}_e$, then $\alpha f + \beta g \in \mathcal{V}_e$ for all $\alpha, \beta \in \mathbb{R}$, because $(\alpha f + \beta g)(-x) = \alpha f(-x) + \beta g(-x) = \alpha f(x) + \beta g(x) = (\alpha f + \beta g)(x)$, thus $\mathcal{V}_e$ is a subspace of $\mathcal{V}$. Similarly, if $f, g \in \mathcal{V}_o$, then $\alpha f + \beta g \in \mathcal{V}_o$ for all $\alpha, \beta \in \mathbb{R}$, because $(\alpha f + \beta g)(-x) = \alpha f(-x) + \beta g(-x) = -\alpha f(x) - \beta g(x) = -(\alpha f + \beta g)(x)$, thus $\mathcal{V}_e$ is a subspace of $\mathcal{V}$.

2. Let $f(x) \in \mathcal{V}$. Define $F(x) = \frac{f(x)+f(-x)}{2}, G(x) = \frac{f(x)-f(-x)}{2}$. Then $F(-x) = F(x) \Rightarrow F \in \mathcal{V}_e$, $G(x) = -G(x) \Rightarrow G \in \mathcal{V}_o$ and $f(x) = F(x) + G(x)$. Thus $\mathcal{V}_e + \mathcal{V}_o = \mathcal{V}$.

3. If $f \in \mathcal{V}_e \cap \mathcal{V}_o$, then $f(-x) = f(x) \because f \in \mathcal{V}_e, f(-x) = -f(x) \because f \in \mathcal{V}_o$. Thus $2f(-x) = 0$ for all $x \in \mathbb{R}$, so $f = 0 \Rightarrow \mathcal{V}_e \cap \mathcal{V}_o = \{0\}$.

∎

Question 4(b) *Find the dimension and basis of the solution space S of the system*

$$\begin{aligned} x_1 + 2x_2 + 2x_3 - x_4 + 3x_5 &= 0 \\ x_1 + 2x_2 + 3x_3 + x_4 + x_5 &= 0 \\ 3x_1 + 6x_2 + 8x_3 + x_4 + 5x_5 &= 0 \end{aligned}$$

Solution.

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 3 & 6 & 8 & 1 & 5 \end{pmatrix} \sim \begin{pmatrix} 1 & 2 & 2 & -1 & 3 \\ 1 & 2 & 3 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

by performing $R_3 - R_1 - 2R_2$.

Thus rank $\mathbf{A} < 3$. Actually rank $\mathbf{A} = 2$, because if $\mathbf{A} = (C_1, C_2, C_3, C_4, C_5)$, where C_i are columns, then C_1 and C_3 are linearly independent.

Adding the first two equations, we get $4x_5 = -2x_1 - 4x_2 - 5x_3$. Subtracting 3 times the second from the first, we get $4x_4 = -2x_1 - 4x_2 - 7x_3$. From these we can see that $\mathbf{X_1} = (2, 0, 0, -1, -1), \mathbf{X_3} = (0, 1, 0, -1, -1), \mathbf{X_3} = (0, 0, 4, -5, -7)$ are three independent solutions. Since rank $\mathbf{A} = 2$, the dimension of the solution space S is 3, and $\{\mathbf{X_1}, \mathbf{X_2}, \mathbf{X_3}\}$ is its basis. ∎

Question 4(c) *Let $\mathcal{W}_1$ and $\mathcal{W}_2$ be subspaces of a finite dimensional vector space $\mathcal{V}$. Prove that $(\mathcal{W}_1 + \mathcal{W}_2)^0 = \mathcal{W}_1^0 \cap \mathcal{W}_2^0$.*

Solution. Let $\mathcal{V}^*$ be the dual of $\mathcal{V}$ i.e. $\mathcal{V}^* = \{f \mid f : \mathcal{V} \longrightarrow \mathbb{R},\ f \text{ linear}\}$. Then $\mathcal{W}^0 = \{f \mid f \in \mathcal{V}^*, \forall \mathbf{w} \in \mathcal{W}. f(\mathbf{w}) = \mathbf{0}\}$. $\mathcal{W}^0$ is a vector subspace of $\mathcal{V}^*$ of dimension $\dim \mathcal{V} - \dim \mathcal{W}$.

If $\mathcal{W}_1 \subseteq \mathcal{W}_2$, then $\mathcal{W}_2^0 \subseteq \mathcal{W}_1^0$, because if $f \in \mathcal{W}_2^0, f(\mathbf{w}) = \mathbf{0} \forall \mathbf{w} \in \mathcal{W}_\mathbf{2}$, and therefore $f(\mathbf{w}) = \mathbf{0} \forall \mathbf{w} \in \mathcal{W}_\mathbf{1}$, so $f \in \mathcal{W}_1^0$.

Now $\mathcal{W}_1 \subseteq \mathcal{W}_1 + \mathcal{W}_2$ and $\mathcal{W}_2 \subseteq \mathcal{W}_1 + \mathcal{W}_2$, so $(\mathcal{W}_1 + \mathcal{W}_2)^0 \subseteq \mathcal{W}_1^0$ and $(\mathcal{W}_1 + \mathcal{W}_2)^0 \subseteq \mathcal{W}_2^0$, thus $(\mathcal{W}_1 + \mathcal{W}_2)^0 \subseteq \mathcal{W}_1^0 \cap \mathcal{W}_2^0$.

Conversely, if $f \in \mathcal{W}_1^0 \cap \mathcal{W}_2^0$, then $f(\mathbf{w}_1) = 0, f(\mathbf{w}_2) = 0$ for all $\mathbf{w}_1 \in \mathcal{W}_1, \mathbf{w}_2 \in \mathcal{W}_2$. Now any $\mathbf{w} \in \mathcal{W}_1 + \mathcal{W}_2$ is of the form $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, so $f(\mathbf{w}) = f(\mathbf{w}_1) + f(\mathbf{w}_2) = 0$, because f is linear. Thus $f \in (\mathcal{W}_1 + \mathcal{W}_2)^0$.

Thus $(\mathcal{W}_1 + \mathcal{W}_2)^0 = \mathcal{W}_1^0 \cap \mathcal{W}_2^0$. ∎

Question 5(a) *Let $\mathbf{H} = \begin{pmatrix} 1 & 1+i & 2i \\ 1-i & 4 & 2-3i \\ -2i & 2+3i & 7 \end{pmatrix}$. Find $\mathbf{P}$ so that $\mathbf{P}'\mathbf{H}\overline{\mathbf{P}}$ is diagonal. Find the signature of $\mathbf{H}$.*

Solution.

$$\begin{pmatrix} 1 & 1+i & 2i \\ 1-i & 4 & 2-3i \\ -2i & 2+3i & 7 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \mathbf{H} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Subtracting $(1-i)R_1$ from R_2, and adding $2iR_1$ to R_3, we get

$$\begin{pmatrix} 1 & 1+i & 2i \\ 0 & 2 & -5i \\ 0 & 5i & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1+i & 1 & 0 \\ 2i & 0 & 1 \end{pmatrix} \mathbf{H} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Subtracting $(1+i)C_1$ from C_2, and adding $-2iC_1$ to C_3, we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & -5i \\ 0 & 5i & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1+i & 1 & 0 \\ 2i & 0 & 1 \end{pmatrix} \mathbf{H} \begin{pmatrix} 1 & -1-i & -2i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Subtracting $\frac{5}{2}iR_2$ from R_3, and adding $\frac{5}{2}iC_2$ to C_3 we get

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -\frac{19}{2} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -1+i & 1 & 0 \\ \frac{5}{2}+\frac{9}{2}i & -\frac{5}{2}i & 1 \end{pmatrix} \mathbf{H} \begin{pmatrix} 1 & -1-i & \frac{5}{2}-\frac{9}{2}i \\ 0 & 1 & \frac{5}{2}i \\ 0 & 0 & 1 \end{pmatrix}$$

Thus $\mathbf{P} = \begin{pmatrix} 1 & -1+i & \frac{5}{2}+\frac{9}{2}i \\ 0 & 1 & -\frac{5}{2}i \\ 0 & 0 & 1 \end{pmatrix}$

Index $=$ Number of positive entries $= 2$. Signature $=$ Number of positive entries - Number of negative entries $= 1$. ∎

Question 5(b) *Prove that every matrix is a root of its characteristic polynomial.*

Solution. This is the Cayley Hamilton theorem, proved in Question 5(a), 1987. ∎

Question 5(c) *If* $\mathbf{B} = \mathbf{AP}$, *where* $\mathbf{P}$ *is nonsingular and* $\mathbf{A}$ *orthogonal, show that* $\mathbf{PB}^{-1}$ *is orthogonal.*

Solution. $\mathbf{B}^{-1} = \mathbf{P}^{-1}\mathbf{A}^{-1}$, so $\mathbf{PB}^{-1} = \mathbf{PP}^{-1}\mathbf{A}^{-1} = \mathbf{A}^{-1}$. Now $(\mathbf{A}^{-1})'\mathbf{A}^{-1} = (\mathbf{AA}')^{-1} = \mathbf{I}$. Similarly $\mathbf{A}^{-1}(\mathbf{A}^{-1})' = (\mathbf{A}'\mathbf{A})^{-1} = \mathbf{I}$, so $\mathbf{PB}^{-1}$ is orthogonal. ∎