UPSC Civil Services Main 1986 - Mathematics
Linear Algebra

Brij Bhooshan

Asst. Professor

B.S.A. College of Engg & Technology

Mathura

Question 1(a) *If* $\mathbf{A}, \mathbf{B}, \mathbf{C}$ *are three* $n \times n$ *matrices, show that* $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$*. Show by an example that matrix multiplication is non-commutative.*

Solution. Let $\mathbf{A} = (a_{ij}), \mathbf{B} = (b_{ij}), \mathbf{C} = (c_{ij})$. Let $\mathbf{BC} = (\beta_{ij}), \mathbf{AB} = (\alpha_{ij})$. Then the ij-th element of the RHS $= \sum_{k=1}^n \alpha_{ik} c_{kj}$. But $\alpha_{ik} = \sum_{l=1}^n a_{il} b_{lk}$, so the ij-th element of the RHS $= \sum_{k=1}^n \sum_{l=1}^n a_{il} b_{lk} c_{kj}$.

Similarly, the ij-th element of the LHS $= \sum_{l=1}^n a_{il}\beta_{lj} = \sum_{l=1}^n a_{il} \sum_{k=1}^n b_{lk} c_{kj}$. Thus $\mathbf{A}(\mathbf{BC}) = (\mathbf{AB})\mathbf{C}$.

Let $\mathbf{A} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then $\mathbf{AB} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$. But $\mathbf{BA} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$. Thus $\mathbf{AB} \neq \mathbf{BA}$. ∎

Question 1(b) *Examine the correctness or otherwise of the following statements:*

1. *The division law is not valid in matrix algebra.*
2. *If* $\mathbf{A}, \mathbf{B}$ *are square matrices each of order* n*, and* $\mathbf{I}$ *is the corresponding unit matrix, then the equation*
$$\mathbf{AB} - \mathbf{BA} = \mathbf{I}$$
can never hold.

Solution.

1. True. Let $\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \mathbf{B} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \mathbf{C} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$. Then $\mathbf{AB} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = \mathbf{AC}$, but $\mathbf{B} \neq \mathbf{C}$.

2. True. We have proved in 1987 question 5(c) that $\mathbf{AB}$ and $\mathbf{BA}$ have the same eigenvalues. Trace of $\mathbf{AB}-\mathbf{BA}$ = trace of $\mathbf{AB}$ - trace of $\mathbf{BA}$ = sum of the eigenvalues of $\mathbf{AB}$ - sum of the eigenvalues of $\mathbf{BA} = 0$. But trace of I_n = n, thus $\mathbf{AB}-\mathbf{BA}=\mathbf{I}$ can never hold.

∎

Question 1(c) *Find a* 3×3 *matrix* $\mathbf{X}$ *such that*

$$\begin{pmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{pmatrix}\mathbf{X} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$$

Solution. $|\mathbf{A}| = \begin{vmatrix} 1 & 2 & -2 \\ -1 & 3 & 0 \\ 0 & -2 & 1 \end{vmatrix} = \mathbf{3-2(-1)-2(2)=1}$, so $\mathbf{A}$ is non-singular. Hence $\mathbf{X} = \mathbf{A}^{-1}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. A simple calculation gives $\mathbf{A}^{-1} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}$. Thus

$$\mathbf{X} = \begin{pmatrix} 3 & 2 & 6 \\ 1 & 1 & 2 \\ 2 & 2 & 5 \end{pmatrix}\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 8 & 9 \\ 1 & 3 & 3 \\ 2 & 7 & 7 \end{pmatrix}$$

. ∎

Question 2(a) *If* $\mathcal{M},\mathcal{N}$ *are two subspaces of a vector space* $\mathcal{S}$, *then show that their dimensions satisfy*

$$\dim\mathcal{M}+\dim\mathcal{N} = \dim(\mathcal{M}\cap\mathcal{N}) + \dim(\mathcal{M}+\mathcal{N})$$

Solution. See 1998 question 1(b). ∎

Question 2(b) *Find a maximal linearly independent subsystem of the system of vectors* $\mathbf{v}_1=(2,-2,-4), \mathbf{v}_2=(1,9,3), \mathbf{v}_3=(-2,-4,1), \mathbf{v}_4=(3,7,-1)$.

Solution. $\mathbf{v}_1,\mathbf{v}_2$ are linearly independent because $a\mathbf{v}_1+b\mathbf{v}_2 = (2a+b,-2a+9b,-4a+3b) = \mathbf{0} \Rightarrow a=b=0$.

$\mathbf{v}_3$ is dependent on $\mathbf{v}_1,\mathbf{v}_2$. If $\mathbf{v}_3 = a\mathbf{v}_1+b\mathbf{v}_2$, then $(2a+b,-2a+9b,-4a+3b) = (-2,-4,1) \Rightarrow a=-\frac{7}{10}, b=-\frac{6}{10}$.

Similarly $\mathbf{v}_4$ is dependent on $\mathbf{v}_1,\mathbf{v}_2$. If $\mathbf{v}_4 = a\mathbf{v}_1+b\mathbf{v}_2$, then $(2a+b,-2a+9b,-4a+3b) = (3,7,-1) \Rightarrow a=b=1$.

Thus the maximally linearly independent set is $\{\mathbf{v}_1,\mathbf{v}_2\}$. ∎

Question 2(c) *Show that the system of equations*

$$\begin{aligned} 4x + y - 2z + w &= 3 \\ x - 2y - z + 2w &= 2 \\ 2x + 5y - w &= -1 \\ 3x + 3y - z - 3w &= 1 \end{aligned}$$

although consistent is not uniquely solvable. Determine a general solution using x as a parameter.

Solution. The coefficient matrix $\mathbf{A} = \begin{pmatrix} 4 & 1 & -2 & 1 \\ 1 & -2 & -1 & 2 \\ 2 & 5 & 0 & -1 \\ 3 & 3 & -1 & -3 \end{pmatrix}$.

The augmented matrix $\mathbf{B} = \begin{pmatrix} 4 & 1 & -2 & 1 & 3 \\ 1 & -2 & -1 & 2 & 2 \\ 2 & 5 & 0 & -1 & -1 \\ 3 & 3 & -1 & -3 & 1 \end{pmatrix}$.

Add R_2 to R_4, and subtract R_1, to get

$$\mathbf{A} \sim \begin{pmatrix} 4 & 1 & -2 & 1 \\ 1 & -2 & -1 & 2 \\ 2 & 5 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \mathbf{B} \sim \begin{pmatrix} 4 & 1 & -2 & 1 & 3 \\ 1 & -2 & -1 & 2 & 2 \\ 2 & 5 & 0 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Since $\begin{vmatrix} 1 & -2 & 1 \\ -2 & -1 & 2 \\ 5 & 0 & -1 \end{vmatrix} = 1 - 16 + 5 \neq 0$, it follows that rank $\mathbf{A} =$ rank $\mathbf{B} = 3$, so the system is consistent. Since rank $\mathbf{A} = 3$, the space of solutions is of dimension 1.

Subtracting the second equation from the fourth, we get $2x + 5y - 5w = -1$. But $2x + 5y - w = -1$, so $w - 5w = 0 \Rightarrow w = 0$.

Now $y - 2z = 3 - 4x, -2y - z = 2 - x \Rightarrow -5z = 8 - 9x \Rightarrow z = \frac{9x-8}{5}$. Now $y = 3 - 4x + 2\frac{9x-8}{5} = \frac{-2x-1}{5}$. Thus the space of solutions is $(x, \frac{-2x-1}{5}, \frac{9x-8}{5}, 0)$. The system does not have a unique solution. ∎

Question 3(a) *Show that every square matrix satisfies its characteristic equation. Using this result or otherwiseshow that if*

$$\mathbf{A} = \begin{pmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

then $\mathbf{A}^4 - 2\mathbf{A}^3 - 2\mathbf{A}^2 + 6\mathbf{A} - 2\mathbf{I} = \mathbf{A}$, where $\mathbf{I}$ is the 3×3 identity matrix.

Solution. The first part is the Cayley Hamilton theorem. See 1987 Question 5(a).

The characteristic equation of $\mathbf{A}$ is $|\mathbf{A} - x\mathbf{I}| = 0$, thus

$$\begin{vmatrix} 1-x & 0 & 2 \\ 0 & -1-x & 1 \\ 0 & 1 & -x \end{vmatrix} = (1-x)(x^2+x-1) = -x^3+2x-1 = 0$$

By the Cayley Hamilton Theorem, $\mathbf{A}^3 - 2\mathbf{A} + \mathbf{I} = \mathbf{0}$.

Thus $\mathbf{A}^4 = 2\mathbf{A}^2 - \mathbf{A}$, and $2\mathbf{A}^3 = 4\mathbf{A} - 2\mathbf{I}$. Hence $\mathbf{A}^4 - 2\mathbf{A}^3 - 2\mathbf{A}^2 + 6\mathbf{A} - 2\mathbf{I} = 2\mathbf{A}^2 - \mathbf{A} - 4\mathbf{A} + 2\mathbf{I} - 2\mathbf{A}^2 + 6\mathbf{A} - 2\mathbf{I} = \mathbf{A}$ as required. ∎

Question 3(b) *1. Show that a square matrix is singular if and only if at least one of its eigenvalues is 0.*

2. The rank of an $n \times n$ matrix $\mathbf{A}$ remains unchanged if it is premultiplied or postmultiplied by a nonsingular matrix, and that $\operatorname{rank}(\mathbf{X}\mathbf{A}\mathbf{X}^{-1}) = \operatorname{rank}(\mathbf{A})$.

Solution.

1. The characteristic polynomial of $\mathbf{A}$ is $|\mathbf{A} - x\mathbf{I}|$. Putting $x = 0$, we see that the constant term in the characteristic polynomial is $|\mathbf{A}|$. Thus if $\mathbf{A}$ has 0 as an eigenvalue iff 0 is a root of the characteristic polynomial iff $|\mathbf{A}| = 0$.

2. Let $\mathbf{A} = \begin{pmatrix} \mathbf{R}_1 \\ \vdots \\ \mathbf{R}_m \end{pmatrix}$, where each $\mathbf{R}_i$ is $1 \times n$, i.e. $\mathbf{A}$ is $m \times n$. Now $\operatorname{rank}(\mathbf{A})$ is the dimension of the row space of $\mathbf{A}$, i.e. the space generated by $\mathbf{R}_1, \ldots, \mathbf{R}_m$. Let $\mathbf{P} = (p_{ij})$ be an $m \times m$ nonsingular matrix. Then $\mathbf{B} = \mathbf{P}\mathbf{A} = \begin{pmatrix} p_{11}\mathbf{R}_1 + p_{12}\mathbf{R}_2 + \ldots + p_{1m}\mathbf{R}_m \\ p_{21}\mathbf{R}_1 + p_{22}\mathbf{R}_2 + \ldots + p_{2m}\mathbf{R}_m \\ \vdots \\ p_{m1}\mathbf{R}_1 + p_{m2}\mathbf{R}_2 + \ldots + p_{mm}\mathbf{R}_m \end{pmatrix}$.
   Thus the rows of $\mathbf{P}\mathbf{A} \subset$ the row space of $\mathbf{A}$, being linear combinations of rows of $\mathbf{A}$. Writing $\mathbf{A} = \mathbf{P}^{-1}\mathbf{B}$, we get that the row space of $\mathbf{A} \subset$ the row space of $\mathbf{B}$, so $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{B})$.

   Let $\mathbf{Q}$ be non-singular $n \times n$, and $\mathbf{C} = \mathbf{A}\mathbf{Q}$. It can be proved as above that the column space of $\mathbf{A}$ = the column space of $\mathbf{C}$, thus $\operatorname{rank}(\mathbf{A}) = \operatorname{rank}(\mathbf{C})$.

   Now by using the above results, $\operatorname{rank}(\mathbf{X}\mathbf{A}\mathbf{X}^{-1}) = \operatorname{rank}(\mathbf{X}\mathbf{A}) = \operatorname{rank}(\mathbf{A})$.

∎

Paper II

Question 4(a) *If $\mathcal{V}_1$ and $\mathcal{V}_2$ are subspaces of a vector space $\mathcal{V}$, then show that* $\dim(\mathcal{V}_1 + \mathcal{V}_2) = \dim(\mathcal{V}_1) + \dim(\mathcal{V}_2) - \dim(\mathcal{V}_1 \cap \mathcal{V}_2)$.

Solution. See 1998, question 1(b). ∎

Question 4(b) *Let $\mathcal{V}$ and $\mathcal{W}$ be vector spaces over the same field F and $\dim \mathcal{V} = n$. Let $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ be a basis of $\mathcal{V}$. Show that a map $f : \{\mathbf{e}_1, \dots, \mathbf{e}_n\} \longrightarrow \mathcal{W}$, can be uniquely extended to a linear transformation $T : \mathcal{V} \longrightarrow \mathcal{W}$ whose restriction to the given basis is f i.e. $T(\mathbf{e}_i) = f(\mathbf{e}_i)$.*

Solution. If $\mathbf{v} = \sum_{i=1}^n a_i \mathbf{e}_i$, define $T(\mathbf{v}) = \sum_{i=1}^n a_i f(\mathbf{e}_i)$. Clearly $T(\mathbf{e}_i) = f(\mathbf{e}_i)$. If $\mathbf{v} = \sum_{i=1}^n a_i \mathbf{e}_i$, $\mathbf{w} = \sum_{i=1}^n b_i \mathbf{e}_i$, then

$$\begin{aligned} T(\alpha\mathbf{v} + \beta\mathbf{w}) &= T(\sum_{i=1}^n (\alpha a_i + \beta b_i)\mathbf{e}_i) \\ &= \sum_{i=1}^n (\alpha a_i + \beta b_i) f(\mathbf{e}_i) \\ &= \alpha \sum_{i=1}^n a_i f(\mathbf{e}_i) + \beta \sum_{i=1}^n b_i f(\mathbf{e}_i) \\ &= \alpha T(\mathbf{v}) + \beta T(\mathbf{w}) \end{aligned}$$

Thus T is a linear transformation.

If U is any other linear transformation satisfying $U(\mathbf{e}_i) = f(\mathbf{e}_i)$, then for any $\mathbf{v} = \sum_{i=1}^n a_i \mathbf{e}_i$, by linearity, $T(\mathbf{v}) = T(\sum_{i=1}^n a_i \mathbf{e}_i) = \sum_{i=1}^n a_i T(\mathbf{e}_i) = \sum_{i=1}^n a_i f(\mathbf{e}_i) = \sum_{i=1}^n a_i U(\mathbf{e}_i) = U(\mathbf{v})$. Since this is true for every $\mathbf{v}$, we have $T = U$. ∎

Question 5(a)

1. *If A and B are two linear transformations and if A^{-1} and B^{-1} exist, show that $(AB)^{-1}$ exists and $(AB)^{-1} = B^{-1}A^{-1}$.*

2. *Prove that similar matrices have the same characteristic polynomial and hence the same eigenvalues.*

3. *Prove that the eigenvalues of a Hermitian matrix are real.*

Solution.

1. Clearly $(AB)(B^{-1}A^{-1}) = AIA^{-1} = AA^{-1} = I$, $(B^{-1}A^{-1})(AB) = B^{-1}A^{-1}AB = B^{-1}B = I$. Thus AB is invertible and its inverse is $B^{-1}A^{-1}$.

2. If $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$ then $|\lambda\mathbf{I} - \mathbf{B}| = |\lambda\mathbf{P}^{-1}\mathbf{P} - \mathbf{P}^{-1}\mathbf{A}\mathbf{P}| = |\mathbf{P}^{-1}||\lambda\mathbf{I} - \mathbf{A}||\mathbf{P}| = |\lambda\mathbf{I} - \mathbf{A}|$. Thus $\mathbf{A}$ and $\mathbf{B}$ have the same characteristic polynomial and therefore the same eigenvalues.

3. See 1993 question 2(c).

∎

Question 5(b) *Reduce* $2x^2 + 4xy + 5y^2 + 4x + 13y - \frac{1}{4} = 0$ *to canonical form.*

Solution.

$$\begin{aligned}
LHS &= 2(x+y+1)^2 - 2y^2 - 2 + 5y^2 + 9y - \frac{1}{4} \\
&= 2(x+y+1)^2 + 3(y^2+3y) - \frac{9}{4} \\
&= 2(x+y+1)^2 + 3(y+\frac{3}{2})^2 - \frac{27}{4} - \frac{9}{4} \\
&= 2X^2 + 3Y^2 - 9 \qquad\qquad \text{where } X = x+y+1, Y = y + \frac{3}{2}
\end{aligned}$$

$2X^2 + 3Y^2 - 9 = 0 \Rightarrow \frac{X^2}{9/2} + \frac{Y^2}{3} = 1$. Thus the given equation is an ellipse. ∎

Question 5(c) *Find the reciprocal of the matrix* $\mathbf{T} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. *Then show that the transform of the matrix* $\mathbf{A} = \frac{1}{2}\begin{pmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{pmatrix}$ *by* $\mathbf{T}$ *i.e.* $\mathbf{TAT}^{-1}$ *is a diagonal matrix. Determine the eigenvalues of the matrix* $\mathbf{A}$.

Solution. $|\mathbf{T}| = -1(-1) + 1(1) = 2$. So

$$\mathbf{T}^{-1} = \frac{1}{2}\begin{pmatrix} A_{11} & A_{21} & A_{31} \\ A_{12} & A_{22} & A_{32} \\ A_{13} & A_{23} & A_{33} \end{pmatrix} = \frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix}$$

Here A_{ij} denotes the cofactor of a_{ij}. Now

$$\begin{aligned}
\mathbf{TAT}^{-1} &= \frac{1}{2}\begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\begin{pmatrix} b+c & c-a & b-a \\ c-b & c+a & a-b \\ b-c & a-c & a+b \end{pmatrix}\frac{1}{2}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 0 & 2a & 2a \\ 2b & 0 & 2b \\ 2c & 2c & 0 \end{pmatrix}\begin{pmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{pmatrix} \\
&= \frac{1}{4}\begin{pmatrix} 4a & 0 & 0 \\ 0 & 4b & 0 \\ 0 & 0 & 4c \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}
\end{aligned}$$

Thus $\mathbf{TAT}^{-1}$ is diagonal. Now the eigenvalues of $\mathbf{A}$ and $\mathbf{TAT}^{-1}$ are the same, so the eigenvalues of $\mathbf{A}$ are a, b, c. ∎