UPSC Civil Services Main 1987 - Mathematics
Linear Algebra

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Mathura

Question 1(a) *1. Find all the matrices which commute with the matrix* $\begin{pmatrix} 7 & -3 \\ 5 & -2 \end{pmatrix}$.

2. Prove that the product of two $n \times n$ symmetric matrices is a symmetric matrix if and only if the matrices commute.

Solution.

1.
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 7 & -3 \\ 5 & -2 \end{pmatrix} = \begin{pmatrix} 7 & -3 \\ 5 & -2 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\Rightarrow \quad \begin{aligned} 7a + 5b &= 7a - 3c && (i) \\ -3a - 2b &= 7b - 3d && (ii) \\ 7c + 5d &= 5a - 2c && (iii) \\ -3c - 2d &= 5b - 2d && (iv) \end{aligned}$$

(i) and (iv) $\Rightarrow 5b = -3c$. From (ii) we get $d = a + 3b$, and from (iii) we get the same thing: $5a - 9c = 5a + 15b = 5d$, or $d = a + 3b$. Thus the required matrices are $\begin{pmatrix} a & b \\ -\frac{5}{3}b & a + 3b \end{pmatrix}$, a, b arbitrary.

2. Given $\mathbf{A}' = \mathbf{A}, \mathbf{B}' = \mathbf{B}$. Suppose $\mathbf{AB} = \mathbf{BA}$, then $(\mathbf{AB})' = \mathbf{B}'\mathbf{A}' = \mathbf{BA} = \mathbf{AB} \Rightarrow \mathbf{AB}$ is symmetric. Let $\mathbf{AB}$ be symmetric. Then $\mathbf{AB} = (\mathbf{AB})' = \mathbf{B}'\mathbf{A}' = \mathbf{BA}$, so $\mathbf{A}$ and $\mathbf{B}$ commute. Thus $\mathbf{AB}$ is symmetric $\Leftrightarrow \mathbf{AB} = \mathbf{BA}$ when $\mathbf{A}' = \mathbf{A}, \mathbf{B}' = \mathbf{B}$.

∎

Question 1(b) *Show that the rank of the product of two square matrices* $\mathbf{A}, \mathbf{B}$ *each of order* n *satisfies the inequality*

$$r_{\mathbf{A}} + r_{\mathbf{B}} - n \leq r_{\mathbf{AB}} \leq \min(r_{\mathbf{A}}, r_{\mathbf{B}})$$

where $r_{\mathbf{C}}$ *stands for the rank of* $\mathbf{C}$*, a square matrix.*

Solution. There exists a non-singular matrix $\mathbf{P}$ such that $\mathbf{PA} = \begin{pmatrix}\mathbf{G}\\ \mathbf{0}\end{pmatrix}$, where $\mathbf{G}$ is a $r_{\mathbf{A}} \times n$ matrix of rank $r_{\mathbf{A}}$. Now $\mathbf{PAB} = \begin{pmatrix}\mathbf{G}\\ \mathbf{0}\end{pmatrix}\mathbf{B}$ has at most $r_{\mathbf{A}}$ non-zero rows obtained on multiplying $r_{\mathbf{A}}$ non-zero rows of $\mathbf{G}$ with $\mathbf{B}$. Thus $r_{\mathbf{PAB}}$, which is the same as rank $r_{\mathbf{AB}}$ as $\mathbf{P}$ is non-singular, $\leq r_{\mathbf{A}}$.

Similarly there exists a non-singular matrix $\mathbf{Q}$ such that $\mathbf{BQ} = \begin{pmatrix}\mathbf{H} & \mathbf{0}\end{pmatrix}$, where $\mathbf{H}$ is a $n \times r_{\mathbf{B}}$ matrix of rank $r_{\mathbf{B}}$. Now $\mathbf{ABQ} = \mathbf{A}\begin{pmatrix}\mathbf{H} & \mathbf{0}\end{pmatrix}$ has at most $r_{\mathbf{B}}$ non-zero, columns, so $r_{\mathbf{ABQ}} \leq r_{\mathbf{B}}$. Now $r_{\mathbf{ABQ}} = r_{\mathbf{AB}}$ as $|\mathbf{Q}| \neq 0$, so $r_{\mathbf{AB}} \leq r_{\mathbf{B}}$, hence $r_{\mathbf{AB}} \leq \min(r_{\mathbf{A}}, r_{\mathbf{B}})$.

Let $S(\mathbf{A})$ denote the space generated by the vectors $\mathbf{r}_1, \dots, \mathbf{r}_n$ where $\mathbf{r}_i$ is the ith row of $\mathbf{A}$, then $\dim(S(\mathbf{A})) = r_{\mathbf{A}}$, similarly $\dim(S(\mathbf{B})) = r_{\mathbf{B}}$. Let S denote the space generated by the rows of $\mathbf{A}$ and $\mathbf{B}$. Clearly $\dim(S) \leq \dim(S(\mathbf{A})) + \dim(S(\mathbf{B})) = r_{\mathbf{A}} + r_{\mathbf{B}}$. Clearly $S(\mathbf{A}+\mathbf{B}) \subseteq S$. Therefore $r_{\mathbf{A}+\mathbf{B}} \leq \dim(S) \leq r_{\mathbf{A}} + r_{\mathbf{B}}$.

Now there exist non-singular matrices $\mathbf{P}, \mathbf{Q}$ such that $\mathbf{PAQ} = \begin{pmatrix}\mathbf{I}_{r_{\mathbf{A}}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}$ or $\mathbf{A} = \mathbf{P}^{-1}\begin{pmatrix}\mathbf{I}_{r_{\mathbf{A}}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{pmatrix}\mathbf{Q}^{-1}$. Let $\mathbf{C} = \mathbf{P}^{-1}\begin{pmatrix}\mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{n-r_{\mathbf{A}}}\end{pmatrix}\mathbf{Q}^{-1}$. Then $\mathbf{A}+\mathbf{C} = \mathbf{P}^{-1}\begin{pmatrix}\mathbf{I}_{r_{\mathbf{A}}} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{n-r_{\mathbf{A}}}\end{pmatrix}\mathbf{Q}^{-1} = \mathbf{P}^{-1}\mathbf{Q}^{-1}$, so $\mathbf{A}+\mathbf{C}$ is nonsingular.

Now rank $\mathbf{B} = \operatorname{rank}((\mathbf{A}+\mathbf{C})\mathbf{B}) \leq \operatorname{rank}(\mathbf{AB}) + \operatorname{rank}(\mathbf{CB})$. But $\operatorname{rank}(\mathbf{CB}) \leq \operatorname{rank}(\mathbf{C}) = n - r_{\mathbf{A}}$. Thus $r_{\mathbf{B}} \leq r_{\mathbf{AB}} + n - r_{\mathbf{A}} \Rightarrow r_{\mathbf{A}} + r_{\mathbf{B}} - n \leq r_{\mathbf{AB}}$. Hence $r_{\mathbf{A}} + r_{\mathbf{B}} - n \leq r_{\mathbf{AB}} \leq \min(r_{\mathbf{A}}, r_{\mathbf{B}})$. ∎

Question 1(c) *If* $1 \leq a \leq 5$*, find the rank of the matrix*

$$\mathbf{A} = \begin{pmatrix}1 & 1 & 1 & 1\\ 1 & 3 & -2 & a\\ 2 & 2a-2 & -a-2 & 3a-1\\ 3 & a+2 & -3 & 2a+1\end{pmatrix}$$

Solution. $|\mathbf{A}| = \begin{vmatrix}1 & 0 & 0 & 0\\ 1 & 2 & -3 & a-1\\ 2 & 2a-4 & -a-4 & 3a-3\\ 3 & a-1 & -6 & 2a-2\end{vmatrix}$ by carrying out the operations $\mathbf{C}_2 - \mathbf{C}_1, \mathbf{C}_3 - \mathbf{C}_1, \mathbf{C}_4 - \mathbf{C}_1$. Thus $|\mathbf{A}| = (a-1)\begin{vmatrix}2 & -3 & 1\\ 2a-4 & -a-4 & 3\\ a-1 & -6 & 2\end{vmatrix} = (a-1)\begin{vmatrix}0 & 0 & 1\\ 2a-10 & -a+5 & 3\\ a-5 & 0 & 2\end{vmatrix} = (a-1)(a-5)^2$.

Thus $|\mathbf{A}| \neq 0$ when $a \neq 1, a \neq 5$. So for $1 < a < 5, \text{rank}\, \mathbf{A} = 4$.

If $a = 5$,

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 5 \\ 2 & 8 & -7 & 14 \\ 3 & 7 & -3 & 11 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & -3 & 4 \\ 2 & 6 & -9 & 12 \\ 3 & 4 & -6 & 8 \end{pmatrix} \qquad (\mathbf{C}_2 - \mathbf{C}_1, \mathbf{C}_3 - \mathbf{C}_1, \mathbf{C}_4 - \mathbf{C}_1) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -3 & 4 \\ 0 & 6 & -9 & 12 \\ 0 & 4 & -6 & 8 \end{pmatrix} \qquad (\mathbf{R}_2 - \mathbf{R}_1, \mathbf{R}_3 - 2\mathbf{R}_1, \mathbf{R}_4 - 3\mathbf{R}_1) \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & -3 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad (\mathbf{R}_3 - 3\mathbf{R}_2, \mathbf{R}_4 - 2\mathbf{R}_2)
\end{aligned}$$

which has rank 2, as $\begin{vmatrix} 1 & 0 \\ 0 & 2 \end{vmatrix} \neq 0$, showing that rank of $\mathbf{A}$ is 2 when $a = 5$.

If $a = 1$,

$$\begin{aligned}
\mathbf{A} &= \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & -2 & 1 \\ 2 & 0 & -3 & 2 \\ 3 & 3 & -3 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 2 & -3 & 0 \\ 2 & -2 & -5 & 0 \\ 3 & 0 & -6 & 0 \end{pmatrix} \qquad (\mathbf{C}_2 - \mathbf{C}_1, \mathbf{C}_3 - \mathbf{C}_1, \mathbf{C}_4 - \mathbf{C}_1)
\end{aligned}$$

which has rank 3 since $\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & -3 \\ 2 & -2 & -5 \end{vmatrix} \neq 0$, showing that rank of $\mathbf{A}$ is 3 when $a = 1$. ∎

Question 2(a) *If the eigenvalues of a matrix* $\mathbf{A}$ *are* $\lambda_j, j = 1, 2, \ldots n$ *and if* $f(x)$ *is a polynomial in* x*, show that the eigenvalues of the polynomial* $f(\mathbf{A})$ *are* $f(\lambda_j), j = 1, 2, \ldots n$.

Solution. Let $\mathbf{x}_r$ be an eigenvector of λ_r. Then $\mathbf{A}^k\mathbf{x}_r = \mathbf{A}^{k-1}(\mathbf{A}\mathbf{x}_r) = \lambda_r\mathbf{A}^{k-1}\mathbf{x}_r = \ldots = \lambda_r^k\mathbf{x}_r$. Thus the eigenvalues of $\mathbf{A}^k$ are $\lambda_j^k, j = 1, 2, \ldots, n$.

Let $f(x) = a_0 + a_1x + \ldots + a_mx^m$. Then $(a_0\mathbf{I} + a_1\mathbf{A} + \ldots + a_m\mathbf{A}^m)\mathbf{x}_r = (a_0 + a_1\lambda_r + \ldots + a_m\lambda_r^m)\mathbf{x}_r = f(\lambda_r)\mathbf{x}_r$. Thus the eigenvalues of $f(\mathbf{A})$ are $f(\lambda_j), j = 1, 2, \ldots n$. ∎

Question 2(b) *If* $\mathbf{A}$ *is skew-symmetric, then show that* $(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A})^{-1}$, *where* $\mathbf{I}$ *is the corresponding identity matrix, is orthogonal.*

Hence construct an orthogonal matrix if $\mathbf{A} = \begin{pmatrix} 0 & \frac{a}{b} \\ -\frac{a}{b} & 0 \end{pmatrix}$.

Solution. For the orthogonality of $(\mathbf{I} - \mathbf{A})(\mathbf{I} + \mathbf{A})^{-1}$, see question 2(a) of 1999.

$\mathbf{I} - \mathbf{A} = \begin{pmatrix} 1 & -\frac{a}{b} \\ \frac{a}{b} & 1 \end{pmatrix}$, and $\mathbf{I} + \mathbf{A} = \begin{pmatrix} 1 & \frac{a}{b} \\ -\frac{a}{b} & 1 \end{pmatrix} \Rightarrow (\mathbf{I} + \mathbf{A})^{-1} = \frac{b}{a^2+b^2}\begin{pmatrix} b & -a \\ a & b \end{pmatrix}$.

Thus $(\mathbf{I}-\mathbf{A})(\mathbf{I}+\mathbf{A})^{-1} = \frac{1}{a^2+b^2}\begin{pmatrix} b & -a \\ a & b \end{pmatrix}\begin{pmatrix} b & -a \\ a & b \end{pmatrix} = \frac{1}{a^2+b^2}\begin{pmatrix} b^2 - a^2 & -2ab \\ 2ab & b^2 - a^2 \end{pmatrix} = \begin{pmatrix} \frac{b^2-a^2}{a^2+b^2} & \frac{-2ab}{a^2+b^2} \\ \frac{2ab}{a^2+b^2} & \frac{b^2-a^2}{a^2+b^2} \end{pmatrix}$, which is the required orthogonal matrix. ∎

Question 2(c)

1. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are arbitrary square matrices of which* $\mathbf{A}$ *is non-singular, show that* $\mathbf{AB}$ *and* $\mathbf{BA}$ *have the same characteristic polynomial.*

2. *Show that a real matrix* $\mathbf{A}$ *is orthogonal if and only if* $|\mathbf{Ax}| = |\mathbf{x}|$ *for all* $\mathbf{x}$.

Solution.

1. $\mathbf{BA} = \mathbf{A^{-1}ABA}$. Thus the characteristic polynomial of $\mathbf{BA}$ is $|\mathbf{x}\mathbf{I} - \mathbf{BA}| = |\mathbf{x}\mathbf{A^{-1}A} - \mathbf{A^{-1}ABA}| = |\mathbf{A}^{-1}||\mathbf{x}\mathbf{I} - \mathbf{AB}||\mathbf{A}| = |\mathbf{x}\mathbf{I} - \mathbf{AB}|$ which is the characteristic polynomial of $\mathbf{AB}$.

2. If $\mathbf{A}$ is orthogonal, i.e. $\mathbf{A}'\mathbf{A} = \mathbf{I}$, then $|\mathbf{Ax}| = \sqrt{\mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x}} = \sqrt{\mathbf{x}'\mathbf{x}} = |\mathbf{x}|$.

   Conversely $|\mathbf{Ax}| = |\mathbf{x}| \Rightarrow \mathbf{x}'\mathbf{A}'\mathbf{A}\mathbf{x} = \mathbf{x}'\mathbf{x} \Rightarrow \mathbf{x}'(\mathbf{A}'\mathbf{A} - \mathbf{I})\mathbf{x} = 0$ for all $\mathbf{x}$. Thus $\mathbf{A}'\mathbf{A} - \mathbf{I} = \mathbf{0}$, so $\mathbf{A}$ is orthogonal.

Note that is $\mathbf{A} = (a_{ij})$ is symmetric, and $\sum_{i,j=1}^n a_{ij}x_ix_j = 0$ for all $\mathbf{x}$, then choose $\mathbf{x} = \mathbf{e_i}$ to get $\mathbf{e_i'Ae_i} = a_{ii} = 0$, and choose $\mathbf{x} = \mathbf{e_i}+\mathbf{e_j}$ to get $0 = \mathbf{x'Ax} = a_{ii}+2a_{ij}+a_{jj} = 2a_{ij} \Rightarrow a_{ij} = 0$. (Here $\mathbf{e_i}$ is the i-th unit vector.) Thus $\mathbf{A} = \mathbf{0}$. ∎

Question 3(a) *Show that a necessary and sufficient condition for a system of linear equations to be consistent is that the rank of the coefficient matrix is equal to the rank of the augmented matrix. Hence show that the system*

$$\begin{aligned} x + 2y + 5z + 9 &= 0 \\ x - y + 3z - 2 &= 0 \\ 3x - 6y - z - 25 &= 0 \end{aligned}$$

is consistent and has a unique solution. Determine this solution.

Solution. Let the system be $\mathbf{A}\mathbf{x} = \mathbf{b}$ where $\mathbf{A}$ is $m \times n$, $\mathbf{x}$ is $n \times 1$ and $\mathbf{b}$ is $m \times 1$. Let $\operatorname{rank} \mathbf{A} = r$. $\mathbf{A} = [\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}]$ where each $\mathbf{c_j}$ is an $m \times 1$ column. We can assume without loss of generality that $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r}$ are linearly independent, $r = \operatorname{rank} \mathbf{A}$. The system is now

$$x_1\mathbf{c_1} + x_2\mathbf{c_2} + \ldots + x_n\mathbf{c_n} = \mathbf{b}$$

, where $\mathbf{x}' = (x_1, \ldots, x_n)$. Suppose $\operatorname{rank}([\mathbf{A}\ \mathbf{b}]) = r$. This means that out of $n+1$ columns, exactly r are independent. But by assumption, $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r}$ are linearly independent, therefore these vectors form a basis for the column space of $[\mathbf{A}\ \mathbf{b}]$. Consequently there exist $\alpha_1, \ldots, \alpha_r$ such that $\alpha_1\mathbf{c_1} + \alpha_2\mathbf{c_2} + \ldots + \alpha_r\mathbf{c_r} = \mathbf{b}$. This gives us the required solution $\{\alpha_1, \ldots, \alpha_r, 0, \ldots, 0\}$ to the linear system.

Conversely, let the system be consistent. Let $\mathbf{A} = [\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}]$ as before, with $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r}$ linearly independent, $r = \operatorname{rank} \mathbf{A}$. Since the column space of $\mathbf{A}$, i.e. the space generated by $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}$ has dimension r, each c_j for $r+1 \leq j \leq n$ is linearly dependent on $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r}$. Since there exist $\alpha_1, \ldots, \alpha_n$ such that $\alpha_1\mathbf{c_1} + \alpha_2\mathbf{c_2} + \ldots + \alpha_n\mathbf{c_n} = \mathbf{b}$, $\mathbf{b}$ is a linear combination of $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}$. But each c_j for $r+1 \leq j \leq n$ is a linear combination of $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r}$, therefore $\mathbf{b}$ is a linear combination of $\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_r}$. Thus the space generated by $\{\mathbf{c_1}, \mathbf{c_2}, \ldots, \mathbf{c_n}, \mathbf{b}\}$ also has dimension r, so $\operatorname{rank}([\mathbf{A}\ \mathbf{b}]) = r = \operatorname{rank} \mathbf{A}$.

The coefficient matrix $\mathbf{A} = \begin{pmatrix} 1 & 2 & 5 \\ 1 & -1 & 3 \\ 3 & -6 & -1 \end{pmatrix}$. $|\mathbf{A}| = 24 \neq 0$, so $\operatorname{rank} \mathbf{A} = 3$. The augmented matrix $\mathbf{B} = \begin{pmatrix} 1 & 2 & 5 & -9 \\ 1 & -1 & 3 & 2 \\ 3 & -6 & -1 & 25 \end{pmatrix}$ has rank ≤ 3, but since $\begin{vmatrix} 1 & 2 & 5 \\ 1 & -1 & 3 \\ 3 & -6 & -1 \end{vmatrix} \neq 0$, it has rank 3. Thus the given system is consistent.

Subtracting the second equation from the first we get $3y + 2z + 11 = 0$. Subtracting 3 times the second equation from the third, we get $3y + 10z + 19 = 0$. Clearly $z = -1, y = -3 \Rightarrow x = 2$. Thus $(2, -3, -1)$ is the unique solution. In fact the only solution of the system is

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{A}^{-1} \begin{pmatrix} -9 \\ 2 \\ 25 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -1 \end{pmatrix}$$

∎

Question 3(b) *In an n-dimensional vector space the system of vectors $\mathbf{x}_j, j = 1, \ldots, r$ are linearly independent and can be expressed linearly in terms of the vectors $\mathbf{y}_k, k = 1, \ldots, s$. Show that $r \leq s$.*

Find a maximal linearly independent subsystem of the linear forms

$$\begin{aligned} f_1 &= x + 2y + z + 3t \\ f_2 &= 4x - y - 5z - 6t \\ f_3 &= x - 3y - 4z - 7t \\ f_4 &= 2x + y - z \end{aligned}$$

Solution. Let $\mathcal{W}$ be the subspace spanned by $\mathbf{y}_k, k = 1, \ldots, s$. Then $\dim \mathcal{W} \leq s$. Since $\mathbf{x}_j \in \mathcal{W}, j = 1, \ldots, r$ because $\mathbf{x}_j$ is a linear combination of $\mathbf{y}_k, k = 1, \ldots, s$, and $\mathbf{x}_j, j = 1, \ldots, r$ are linearly independent, $\dim \mathcal{W} \geq r \Rightarrow r \leq s$.

Clearly f_1 and f_4 are linearly independent. f_2 is linearly expressible in terms of f_1 and f_4 because $f_2 = af_1 + bf_4 \Rightarrow a + 2b = 4, 2a + b = -1, a - b = 5, 3a = -6 \Rightarrow a = -2, b = 3$ satisfy all four, hence $f_2 = -2f_1 + 3f_4$. Similarly $f_3 = -\frac{7}{3}f_1 + \frac{5}{3}f_4$. Thus $\{f_1, f_4\}$ is a maximally independent subsystem. ∎

Paper II

Question 4(a) *Let $T : \mathcal{V} \longrightarrow \mathcal{W}$ be a linear transformation. If $\mathcal{V}$ is finite dimensional, show that*

$$\operatorname{rank} T + \operatorname{nullity} T = \dim \mathcal{V}$$

Solution. See question 1(a) of 1992. ∎

Question 4(b) *Prove that two finite dimensional vector spaces $\mathcal{V}, \mathcal{W}$ over the same field $\mathcal{F}$ are isomorphic if they are of the same dimension n.*

Solution. Let $\dim \mathcal{V} = \dim \mathcal{W} = n$. Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis of $\mathcal{V}$, and $\mathbf{w}_1, \ldots, \mathbf{w}_n$ be a basis of $\mathcal{W}$. Define $T : \mathcal{V} \longrightarrow \mathcal{W}$ by $T(\mathbf{v}_i) = \mathbf{w}_i$ and if $\mathbf{v} \in \mathcal{V}, \mathbf{v} = \sum_{i=1}^n a_i \mathbf{v}_i, a_i \in \mathbb{R}$ then $T(\mathbf{v}) = \sum_{i=1}^n a_i T(\mathbf{v}_i)$. Then

1. T is a linear transformation. If $\mathbf{v} = \sum_{i=1}^n a_i \mathbf{v}_i, \mathbf{u} = \sum_{i=1}^n b_i \mathbf{v}_i$ then

$$\begin{aligned}
T(\alpha \mathbf{v} + \beta \mathbf{u}) &= T(\sum_{i=1}^n (\alpha a_i + \beta b_i) T(\mathbf{v}_i) \\
&= \sum_{i=1}^n (\alpha a_i + \beta b_i) T(\mathbf{v}_i) \\
&= \alpha \sum_{i=1}^n a_i T(\mathbf{v}_i) + \beta \sum_{i=1}^n b_i T(\mathbf{v}_i) \\
&= \alpha T(\mathbf{v}) + \beta T(\mathbf{u})
\end{aligned}$$

2. T is 1-1. Let $T(\mathbf{v}) = \mathbf{0}$, where $\mathbf{v} = \sum_{i=1}^n a_i \mathbf{v}_i$. Then $\mathbf{0} = T(\mathbf{v}) = \sum_{i=1}^n a_i T(\mathbf{v}_i) = \sum_{i=1}^n a_i \mathbf{w}_i \Rightarrow a_i = 0, i = 1, \ldots, n$, because $\mathbf{w}_1, \ldots, \mathbf{w}_n$ are linearly independent. Thus $T(\mathbf{v}) = \mathbf{0} \Rightarrow \mathbf{v} = \mathbf{0}$.

3. T is onto. If $\mathbf{w} \in \mathcal{W}$ and $\mathbf{w} = \sum_{i=0}^n b_i \mathbf{w}_i$, then $T(\mathbf{v}) = \mathbf{w}$ where $\mathbf{v} = \sum_{i=1}^n b_i \mathbf{v}_i$.

∎

Note: The converse of 4(b) is also true i.e. if $T : \mathcal{V} \longrightarrow \mathcal{W}$ is an isomorphism i.e. $\mathcal{V}, \mathcal{W}$ are isomorphic, then $\dim \mathcal{V} = \dim \mathcal{W}$.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be a basis of $\mathcal{V}$. Then $\{\mathbf{w}_1 = T(\mathbf{v}_1), \ldots, \mathbf{w}_n = T(\mathbf{v}_n)\}$ is a basis of $\mathcal{W}$.

$\mathbf{w}_1, \ldots, \mathbf{w}_n$ **are linearly independent**. If $\sum_{i=0}^n b_i \mathbf{w}_i = \mathbf{0}$, then $\sum_{i=0}^n b_i T\mathbf{v}_i = \mathbf{0} \Rightarrow T(\sum_{i=0}^n b_i \mathbf{v}_i) = \mathbf{0} \Rightarrow \sum_{i=0}^n b_i \mathbf{v}_i = \mathbf{0} \Rightarrow b_i = 0$ for $1 \leq i \leq n$, because $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.

$\mathbf{w}_1, \ldots, \mathbf{w}_n$ **generate W**. If $\mathbf{w} \in \mathcal{W}$, then there exists a $\mathbf{v} \in \mathcal{V}$ such that $T(\mathbf{v}) = \mathbf{w}$, because T is onto. Let $\mathbf{v} = \sum_{i=0}^n b_i \mathbf{v}_i$, then $\mathbf{w} = T(\mathbf{v}) = T(\sum_{i=0}^n b_i \mathbf{v}_i) = \sum_{i=0}^n b_i T(\mathbf{v}_i) = \sum_{i=0}^n b_i \mathbf{w}_i$.

Question 5(a) *Prove that every square matrix is the root of its characteristic polynomial.*

Solution. This is the Cayley Hamilton Theorem. Let $\mathbf{A}$ be a matrix of order n. Let

$$|\mathbf{A} - x\mathbf{I}| = a_o + a_1 x + \ldots + a_n x^n$$

Then we wish to show that

$$a_o \mathbf{I} + a_1 \mathbf{A} + \ldots + a_n \mathbf{A}^n = \mathbf{0}$$

Suppose the adjoint of $\mathbf{A} - x\mathbf{I}$ is $\mathbf{B}_0 + \mathbf{B}_1 x + \ldots + \mathbf{B}_{n-1} x^{n-1}$, where $\mathbf{B}_i$ are matrices of order n. Then by definition of the adjoint,

$$(\mathbf{A} - x\mathbf{I})(\mathbf{B}_0 + \mathbf{B}_1 x + \ldots + \mathbf{B}_{n-1} x^{n-1}) = |\mathbf{A} - x\mathbf{I}|\mathbf{I}$$

Substituting for $|\mathbf{A} - x\mathbf{I}|$ the expression $a_o + a_1 x + \ldots + a_n x^n$ and equating coefficients of like powers, we get

$$\begin{aligned}
\mathbf{AB}_0 &= a_o \mathbf{I} \\
\mathbf{AB}_1 - \mathbf{B}_0 &= a_1 \mathbf{I} \\
\mathbf{AB}_2 - \mathbf{B}_1 &= a_2 \mathbf{I} \\
&\ldots \\
\mathbf{AB}_{n-1} - \mathbf{B}_{n-2} &= a_{n-1} \mathbf{I} \\
-\mathbf{B}_{n-1} &= a_n \mathbf{I}
\end{aligned}$$

Multiplying these equations successively by $\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \ldots, \mathbf{A}^n$ on the left and adding, we get $\mathbf{0} = a_o \mathbf{I} + a_1 \mathbf{A} + \ldots + a_n \mathbf{A}^n$, which was to be proved. ∎

Question 5(b) *Determine the eigenvalues and the corresponding eigenvectors of*

$$\mathbf{A} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 3 & 1 \\ 1 & 2 & 2 \end{pmatrix}$$

Solution.

$$|\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda - 2 & -2 & -1 \\ -1 & \lambda - 3 & -1 \\ -1 & -2 & \lambda - 2 \end{vmatrix} = 0$$

$$\begin{aligned} \Rightarrow \quad (\lambda-2)^2(\lambda-3) - 2(\lambda-2) + 2(-\lambda+2) - 2 - 2 - (\lambda-3) &= 0 \\ \Rightarrow \quad (\lambda^2 - 4\lambda + 4)(\lambda - 3) - 5\lambda + 7 &= 0 \\ \Rightarrow \quad \lambda^3 - 7\lambda^2 + 11\lambda - 5 = (\lambda-1)(\lambda^2 - 6\lambda + 5) &= 0 \\ \Rightarrow \quad \lambda = 1, 5, 1 & \end{aligned}$$

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 5$. Then

$$\begin{pmatrix} 3 & -2 & -1 \\ -1 & 2 & -1 \\ -1 & -2 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $3x_1 - 2x_2 - x_3 = 0, -x_1 + 2x_2 - x_3 = 0, -x_1 - 2x_2 + 3x_3 = 0 \Rightarrow x_1 = x_2 = x_3$. Thus $(1, 1, 1)$ is an eigenvector for $\lambda = 5$. In fact (x, x, x) with $x \neq 0$ are eigenvectors for $\lambda = 5$.

Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 1$. Then

$$\begin{pmatrix} -1 & -2 & -1 \\ -1 & -2 & -1 \\ -1 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $x_1 + 2x_2 + x_3 = 0$. We can take $\mathbf{x_1} = (1, 0, -1)$ and $\mathbf{x_2} = (0, 1, -2)$ as eigenvectors for $\lambda = 1$. These are linearly independent, and all eigenvectors for $\lambda = 1$ are linear combinations of $\mathbf{x_1}, \mathbf{x_2}$.

Let $\mathbf{P} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & -1 & -2 \end{pmatrix}$. Then $\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

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Question 5(c) *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be* n *square matrices over* F. *Show that* $\mathbf{AB}$ *and* $\mathbf{BA}$ *have the same eigenvalues.*

Solution. If $\mathbf{A}$ is non-singular, then

$$\mathbf{BA} = \mathbf{A}^{-1}\mathbf{ABA} \Rightarrow |x\mathbf{I} - \mathbf{BA}| = |x\mathbf{A}^{-1}\mathbf{A} - \mathbf{A}^{-1}\mathbf{ABA}| = |\mathbf{A}^{-1}||x\mathbf{I} - \mathbf{AB}||\mathbf{A}| = |x\mathbf{I} - \mathbf{AB}|$$

Thus the characteristic polynomials of $\mathbf{AB}$ and $\mathbf{BA}$ are the same, so they have the same eigenvalues.

If $\mathbf{A}$ is singular, then let $\text{rank}(\mathbf{A}) = \mathbf{r}$. Then there exist $\mathbf{P}, \mathbf{Q}$ non-singular such that $\mathbf{PAQ} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$. Now $\mathbf{PABP}^{-1} = \mathbf{PAQQ}^{-1}\mathbf{BP}^{-1} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \mathbf{Q}^{-1}\mathbf{BP}^{-1}$. Let $\mathbf{Q}^{-1}\mathbf{BP}^{-1} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{pmatrix}$, where $\mathbf{B}_1$ is $r \times r$, $\mathbf{B}_2$ is $r \times n - r$, $\mathbf{B}_3$ is $n - r \times r$, $\mathbf{B}_4$ is $n - r \times n - r$. Then $\mathbf{PABP}^{-1} = \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{pmatrix} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{0} & \mathbf{0} \end{pmatrix}$, so the characteristic roots of $\mathbf{AB}$ are the same as those of $\mathbf{B}_1$, along with 0 repeated $n - r$ times.

Now $\mathbf{Q}^{-1}\mathbf{BAQ} = \mathbf{Q}^{-1}\mathbf{BP}^{-1}\mathbf{PAQ} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{B}_2 \\ \mathbf{B}_3 & \mathbf{B}_4 \end{pmatrix} \begin{pmatrix} \mathbf{I}_r & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{pmatrix} = \begin{pmatrix} \mathbf{B}_1 & \mathbf{0} \\ \mathbf{B}_3 & \mathbf{0} \end{pmatrix}$ so the characteristic roots of $\mathbf{BA}$ are the same as those of $\mathbf{B}_1$, along with 0 repeated $n - r$ times. Thus $\mathbf{BA}$ and $\mathbf{AB}$ have the same characteristic roots. ∎