UPSC Civil Services Main 1989 - Mathematics
Linear Algebra

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Question 1(a) *Find a basis for the null space of the matrix* $\mathbf{A} = \begin{pmatrix} 3 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix}$.

Solution. $\mathbf{A}$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^2$ defined by $\mathbf{A}(\mathbf{e_1}) = 3\mathbf{e_1^*},\ \mathbf{A}(\mathbf{e_2}) = \mathbf{e_1^*} + \mathbf{e_2^*},\ \mathbf{A}(\mathbf{e_3}) = -\mathbf{e_1^*} + 2\mathbf{e_2^*}$, where $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}$ is the standard basis of $\mathbb{R}^3$ and $\mathbf{e_1^*}, \mathbf{e_2^*}$ is the standard basis of $\mathbb{R}^2$. Thus $\mathbf{A}(a, b, c) = \mathbf{e_1^*}(3a + b - c) + \mathbf{e_2^*}(b + 2c)$. Consequently, $(a, b, c) \in$ null space of $\mathbf{A}$ if and only if $3a + b - c = 0, b + 2c = 0 \Rightarrow b = -2c, a = c$. Thus null space of $\mathbf{A}$ is $\{(c, -2c, c) \mid c \in \mathbb{R}\}$. Note that $\operatorname{rank} \mathbf{A} = 2$, so the null space has dimension 1. A basis for the null space is $(1, -2, 1)$, any other multiple of this can also be regarded as a basis. ∎

Question 1(b) *If $\mathcal{W}$ is a subspace of a finite dimensional vector space $\mathcal{V}$ then prove that* $\dim \mathcal{V}/\mathcal{W} = \dim \mathcal{V} - \dim \mathcal{W}$.

Solution. Let $\mathbf{v_1}, \ldots, \mathbf{v_r}$ be a basis of $\mathcal{W}$, $\dim \mathcal{W} = r$. Let $\mathbf{v_{r+1}}, \ldots, \mathbf{v_n}$ be $n-r$ vectors in $\mathcal{V}$ so chosen that $\mathbf{v_1}, \ldots, \mathbf{v_n}$ is a basis of $\mathcal{V}$, $\dim \mathcal{V} = n$. We will show that $\mathbf{v_i} + \mathcal{W}, r+1 \leq i \leq n$ is a basis of $\mathcal{V}/\mathcal{W} \Rightarrow \dim \mathcal{V}/\mathcal{W} = n - r$.

First we show linear independence:

$$\begin{aligned}
& \sum_{i=r+1}^{n} \alpha_i(\mathbf{v_i} + \mathcal{W}) = 0 \\
\Rightarrow\ & \sum_{i=r+1}^{n} \alpha_i \mathbf{v_i} + \mathcal{W} = \mathbf{0} + \mathcal{W} \\
\Rightarrow\ & \sum_{i=r+1}^{n} \alpha_i \mathbf{v_i} \in \mathcal{W} \\
\Rightarrow\ & \sum_{i=r+1}^{n} \alpha_i \mathbf{v_i} = \sum_{i=1}^{r} -\alpha_i \mathbf{v_i} \text{ (say)} \\
\Rightarrow\ & \sum_{i=1}^{n} \alpha_i \mathbf{v_i} = \mathbf{0} \\
\Rightarrow\ & \alpha_i = 0, 1 \leq i \leq n \ (\mathbf{v_i} \text{ are linearly independent.})
\end{aligned}$$

Thus $\mathbf{v_i} + \mathcal{W}, r+1 \leq i \leq n$ are linearly independent.

If $\mathbf{v} + \mathcal{W}$ is any element of $\mathcal{V}/\mathcal{W}$, then $\mathbf{v} = \sum_{i=1}^{n} \alpha_i \mathbf{v_i}$ as $\mathbf{v} \in \mathcal{V}$. Therefore $\mathbf{v} + \mathcal{W} = \sum_{i=1}^{n} \alpha_i \mathbf{v_i} + \mathcal{W} = \sum_{i=1}^{n} \alpha_i(\mathbf{v_i} + \mathcal{W}) = \sum_{i=r+1}^{n} \alpha_i(\mathbf{v_i} + \mathcal{W})$ because $\mathbf{v_1} + \mathcal{W} = \ldots = \mathbf{v_r} + \mathcal{W} = \mathcal{W}$. Thus $\mathbf{v_i} + \mathcal{W}, r+1 \leq i \leq n$ generate $\mathcal{V}/\mathcal{W}$. Hence $\dim \mathcal{V}/\mathcal{W} = n - r = \dim \mathcal{V} - \dim \mathcal{W}$ ∎

Question 1(c) *Show that all vectors* (x_1, x_2, x_3, x_4) *in the vector space* $\mathcal{V}_4(\mathbb{R})$ *which obey* $x_4 - x_3 = x_2 - x_1$ *form a subspace* $\mathcal{V}$*. Show further that* $\mathcal{V}$ *is spanned by* $\xi_1 = (1, 0, 0, -1), \xi_2 = (0, 1, 0, 1), \xi_3 = (0, 0, 1, 1)$.

Solution. If $\mathbf{y} = (y_1, y_2, y_3, y_4), \mathbf{z} = (z_1, z_2, z_3, z_4) \in \mathcal{V}$ then $\alpha\mathbf{y} + \beta\mathbf{z} = (a_1, a_2, a_3, a_4) \in \mathcal{V}$ because

$$\begin{aligned}
a_4 - a_3 &= (\alpha y_4 + \beta z_4) - (\alpha y_3 + \beta z_3) \\
&= \alpha(y_4 - y_3) + \beta(z_4 - z_3) \\
&= \alpha(y_2 - y_1) + \beta(z_2 - z_1) \quad \because y_4 - y_3 = y_2 - y_1, z_4 - z_3 = z_2 - z_1 \\
&= a_2 - a_1
\end{aligned}$$

Thus $\mathcal{V}$ is a subspace of $\mathcal{V}_4(\mathbb{R})$. Note that $\mathcal{V} \neq \emptyset$.

Clearly ξ_1, ξ_2, ξ_3 are linearly independent $\Rightarrow \dim \mathcal{V} \geq 3$. But $\mathcal{V} \neq \mathcal{V}_4(\mathbb{R})$ because $(1, 0, 0, 0) \notin \mathcal{V} \therefore \dim \mathcal{V} < 4 \Rightarrow \dim \mathcal{V} = 3$.

Hence ξ_1, ξ_2, ξ_3 is a basis of $\mathcal{V}$ and therefore span $\mathcal{V}$. ∎

Question 2(a) *Let* $\mathbf{P}$ *be a real skew-symmetric matrix and* $\mathbf{I}$ *the corresponding unit matrix. Show that* $\mathbf{I} - \mathbf{P}$ *is non-singular. Also show that* $\mathbf{Q} = (\mathbf{I} + \mathbf{P})(\mathbf{I} - \mathbf{P})^{-1}$ *is orthogonal.*

Solution. We have proved (question 2(a), year 1998) that the eigenvalues of a skew-Hermitian and therefore of a skew-symmetric matrix are zero or pure imaginary. This means $|\mathbf{I}-\mathbf{P}| \neq 0$ because 1 cannot be an eigenvalue of $\mathbf{P}$.

$\mathbf{Q}'\mathbf{Q} = [(\mathbf{I}-\mathbf{P})^{-1}]'(\mathbf{I}+\mathbf{P})'(\mathbf{I}+\mathbf{P})(\mathbf{I}-\mathbf{P})^{-1} = (\mathbf{I}+\mathbf{P})^{-1}(\mathbf{I}-\mathbf{P})(\mathbf{I}+\mathbf{P})(\mathbf{I}-\mathbf{P})^{-1}$. But $(\mathbf{I}-\mathbf{P})(\mathbf{I}+\mathbf{P}) = \mathbf{I}-\mathbf{P}^2 = (\mathbf{I}+\mathbf{P})(\mathbf{I}-\mathbf{P})$, therefore $\mathbf{Q}'\mathbf{Q} = \mathbf{I}$. Similarly $\mathbf{Q}\mathbf{Q}' = \mathbf{I} \Rightarrow \mathbf{Q}$ is orthogonal. ∎

Related Results:

1. If $\mathbf{S}$ is skew-Hermitian, then $\mathbf{A} = (\mathbf{I}+\mathbf{S})(\mathbf{I}-\mathbf{S})^{-1}$ is unitary. Conversely, if $\mathbf{A}$ is unitary, then $\mathbf{A}$ can be written as $\mathbf{A} = (\mathbf{I}+\mathbf{S})(\mathbf{I}-\mathbf{S})^{-1}$ for some skew-Hermitian matrix $\mathbf{S}$ provided -1 is not an eigenvalue of $\mathbf{A}$.

   Proof:

$$\begin{aligned}
\overline{\mathbf{A}}' &= (\overline{(\mathbf{I}-\mathbf{S})^{-1}})'\overline{(\mathbf{I}+\mathbf{S})}' = (\mathbf{I}-\overline{\mathbf{S}}')^{-1}(\mathbf{I}+\overline{\mathbf{S}}') \\
&= (\mathbf{I}+\mathbf{S})^{-1}(\mathbf{I}-\mathbf{S}) \\
\therefore \mathbf{A}\overline{\mathbf{A}}' &= (\mathbf{I}+\mathbf{S})(\mathbf{I}-\mathbf{S})^{-1}(\mathbf{I}+\mathbf{S})^{-1}(\mathbf{I}-\mathbf{S}) \\
&= (\mathbf{I}+\mathbf{S})(\mathbf{I}+\mathbf{S})^{-1}(\mathbf{I}-\mathbf{S})^{-1}(\mathbf{I}-\mathbf{S}) = \mathbf{I} \\
&\qquad \because (\mathbf{I}-\mathbf{S})^{-1}(\mathbf{I}+\mathbf{S})^{-1} = (\mathbf{I}-\mathbf{S}^2)^{-1} = (\mathbf{I}+\mathbf{S})^{-1}(\mathbf{I}-\mathbf{S})^{-1}
\end{aligned}$$

   Similarly $\overline{\mathbf{A}}'\mathbf{A} = \mathbf{I}$, so $\mathbf{A}$ is unitary.

   Now $\mathbf{A}(\mathbf{I}-\mathbf{S}) = \mathbf{I}+\mathbf{S} \Rightarrow \mathbf{A}-\mathbf{I} = (\mathbf{A}+\mathbf{I})\mathbf{S} \Rightarrow \mathbf{S} = (\mathbf{A}+\mathbf{I})^{-1}(\mathbf{A}-\mathbf{I})$. It can be checked as above that $\mathbf{S}$ is skew-Hermitian. Note that $|\mathbf{A}+\mathbf{I}| \neq 0$.

2. If $\mathbf{H}$ is Hermitian, then $\mathbf{A} = (\mathbf{H}+i\mathbf{I})^{-1}(\mathbf{H}-i\mathbf{I})$ is unitary and every unitary matrix can be thus represented provided it does not have -1 as its eigenvalue.

3. If $\mathbf{S}$ is real, $\mathbf{S}' = -\mathbf{S}$ and $\mathbf{S}^2 = -\mathbf{I}$, then $\mathbf{S}$ is orthogonal and of even order, and there exist non-null vectors $\mathbf{x}, \mathbf{y}$ such that $\mathbf{x}'\mathbf{x} = \mathbf{y}'\mathbf{y} = 1$, $\mathbf{x}'\mathbf{y} = 0$, $\mathbf{S}\mathbf{x}+\mathbf{y} = \mathbf{0}, \mathbf{S}\mathbf{y} = \mathbf{x}$.

   Proof: $\mathbf{S}'\mathbf{S} = -\mathbf{S}\mathbf{S} = \mathbf{I}$, so $\mathbf{S}$ is orthogonal, $|\mathbf{S}| \neq 0 \Rightarrow \mathbf{S}$ is of even order.

   Choose $\mathbf{y}$ such that $\mathbf{y}'\mathbf{y} = 1$. Then $\mathbf{y}'\mathbf{S}\mathbf{y} = (\mathbf{y}'\mathbf{S}\mathbf{y})' = \mathbf{y}'\mathbf{S}'\mathbf{y} = -\mathbf{y}'\mathbf{S}\mathbf{y} \Rightarrow \mathbf{y}'\mathbf{S}\mathbf{y} = 0$. Set $\mathbf{x} = \mathbf{S}\mathbf{y}$, then $\mathbf{y}'\mathbf{x} = 0$, $\mathbf{S}\mathbf{x}+\mathbf{y} = \mathbf{0}$. In addition, $\mathbf{x}'\mathbf{x} = \mathbf{y}'\mathbf{S}'\mathbf{S}\mathbf{y} = \mathbf{y}'\mathbf{y} = 1$.

Question 2(b) *Show that an $n \times n$ matrix $\mathbf{A}$ is similar to a diagonal matrix if and only if the set of eigenvectors of $\mathbf{A}$ includes a set of n linearly independent vectors.*

Solution. See question 2(c) of 1998. ∎

Question 2(c) *Let r_1, r_2 be distinct eigenvalues of a matrix $\mathbf{A}$ and let $\xi_\mathbf{i}$ be an eigenvector corresponding to $r_i, i = 1, 2$. If $\mathbf{A}$ is Hermitian, show that $\overline{\xi_\mathbf{1}}'\xi_\mathbf{2} = 0$.*

Solution. See question 2(c) of 1993. ∎

Question 3(a) *Find the roots of the equation $|x\mathbf{A} - \mathbf{B}| = 0$ where $\mathbf{A} = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 4 \end{smallmatrix}\right)$, and $\mathbf{B} = \left(\begin{smallmatrix} 0 & 3 \\ 3 & 0 \end{smallmatrix}\right)$. Use the result to show that the real quadratic forms $F = x_1^2 + 2x_1x_2 + 4x_2^2$, $G = 6x_1x_2$ can be simultaneously reduced by a non-singular linear substitution to $y_1^2 + y_2^2, y_1^2 - 3y_2^2$.*

Solution. $|x\mathbf{A} - \mathbf{B}| = \begin{vmatrix} x & x-3 \\ x-3 & 4x \end{vmatrix} = 4x^2 - (x-3)^2 \Rightarrow \pm 2x = x - 3 \Rightarrow x = -3, 1.$

Let $\mathbf{x_1} = (x_1, x_2)$ be a row vector such that $(\mathbf{A} - \mathbf{B})\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \mathbf{0}$.

$$\begin{pmatrix} 1 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0} \Rightarrow x_1 - 2x_2 = 0$$

We take $x_1 = 2, x_2 = 1$, so $\mathbf{x_1} = (2, 1)$.

Let $\mathbf{x_2} = (x_1, x_2)$ be a row vector such that $(-3\mathbf{A} - \mathbf{B})\left(\begin{smallmatrix} x_1 \\ x_2 \end{smallmatrix}\right) = \mathbf{0}$.

$$\begin{pmatrix} -3 & -6 \\ -6 & -12 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \mathbf{0} \Rightarrow x_1 + 2x_2 = 0$$

We take $x_1 = -2, x_2 = 1$, so $\mathbf{x_2} = (-2, 1)$.

$\mathbf{x_1 A x_1'} = (2, 1)\left(\begin{smallmatrix} 1 & 1 \\ 1 & 4 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right) = (2, 1)\left(\begin{smallmatrix} 3 \\ 6 \end{smallmatrix}\right) = 12.$
$\mathbf{x_2 A x_2'} = (-2, 1)\left(\begin{smallmatrix} 1 & 1 \\ 1 & 4 \end{smallmatrix}\right)\left(\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right) = (-2, 1)\left(\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right) = 4.$
Note that $\mathbf{x_1 A x_2'} = 0$.
$\mathbf{x_1 B x_1'} = (2, 1)\left(\begin{smallmatrix} 0 & 3 \\ 3 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right) = (2, 1)\left(\begin{smallmatrix} 3 \\ 6 \end{smallmatrix}\right) = 12.$
$\mathbf{x_2 B x_2'} = (-2, 1)\left(\begin{smallmatrix} 0 & 3 \\ 3 & 0 \end{smallmatrix}\right)\left(\begin{smallmatrix} -2 \\ 1 \end{smallmatrix}\right) = (-2, 1)\left(\begin{smallmatrix} 3 \\ -6 \end{smallmatrix}\right) = -12.$
Note that $\mathbf{x_1 B x_2'} = 0$.

Thus if $\mathbf{P} = [\mathbf{x_1}', \mathbf{x_2}']$, then $\mathbf{P'AP} = \left(\begin{smallmatrix} 12 & 0 \\ 0 & 4 \end{smallmatrix}\right)$, and $\mathbf{P'BP} = \left(\begin{smallmatrix} 12 & 0 \\ 0 & -12 \end{smallmatrix}\right)$. Let $\mathbf{Q} = \left(\begin{smallmatrix} \frac{1}{\sqrt{12}} & 0 \\ 0 & \frac{1}{2} \end{smallmatrix}\right)$, then $\mathbf{Q'P'APQ} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ and $\mathbf{Q'P'BPQ} = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -3 \end{smallmatrix}\right)$ as desired. Thus the required non-singular linear transformation is $\mathbf{PQ}$. ∎

Question 3(b) *Show that* $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & \tan\frac{\theta}{2} \\ -\tan\frac{\theta}{2} & 1 \end{pmatrix}^{-1}$.

Solution.

$$\begin{aligned} R.H.S &= \begin{pmatrix} 1 & -\tan\frac{\theta}{2} \\ \tan\frac{\theta}{2} & 1 \end{pmatrix} \begin{pmatrix} \cos^2\frac{\theta}{2} & -\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2}\cos\frac{\theta}{2} & \cos^2\frac{\theta}{2} \end{pmatrix} \\ &= \begin{pmatrix} \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} & -2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} & -\sin^2\frac{\theta}{2} + \cos^2\frac{\theta}{2} \end{pmatrix} = L.H.S \end{aligned}$$

∎

Question 3(c) *Verify the Cayley-Hamilton theorem for* $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$.

Solution. The characteristic equation for $\mathbf{A}$ is $\begin{vmatrix} -\lambda & 1 \\ -2 & 3-\lambda \end{vmatrix} = 0 \Rightarrow -3\lambda + \lambda^2 + 2 = 0$

Thus according to the Cayley-Hamilton theorem $\mathbf{A}^2 - 3\mathbf{A} + 2\mathbf{I} = \mathbf{0}$.

$\mathbf{A}^2 = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} = \begin{pmatrix} -2 & 3 \\ -6 & 7 \end{pmatrix}$

$\begin{pmatrix} -2 & 3 \\ -6 & 7 \end{pmatrix} - 3 \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix} + 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

Thus the Cayley Hamilton theorem is verified for $\mathbf{A} = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$. ■