# UPSC Civil Services Main 1994 - Mathematics
# Linear Algebra

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**Question 1(a)** *Show that* $f_1(t) = 1, f_2(t) = t - 2, f_3(t) = (t-2)^2$ *forms a basis of* $\mathcal{P}_3 = \{\textit{Space of polynomials of degree} \leq 2\}$. *Express* $3t^2 - 5t + 4$ *as a linear combination of* $f_1, f_2, f_3$.

**Solution.** If $\alpha_1 f_1 + \alpha_2 f_2 + \alpha_3 f_3 \equiv 0$, then $\alpha_3$ being the coefficient of $t^2$ is equal to 0. Then coefficient of $t$ is $\alpha_2$ so it must be 0, hence $\alpha_1 = 0$. Thus $f_1, f_2, f_3$ are linearly independent. Since $\{1, t, t^2\}$ is a basis for $\mathcal{P}_3$, its dimension is 3, hence $f_1, f_2, f_3$ is a basis of $\mathcal{P}_3$.

Now by Taylor's expansion $p(t) = 3t^2 - 5t + 4 = p(2) + p'(2)(t-2) + \frac{p"(2)}{2!}(t-2)^2 = 6 + 7(t-2) + 3(t-2)^2 = 6f_1 + 7f_2 + 3f_3$. ∎

**Question 1(b)** *Let* $\mathbf{T} : \mathbb{R}^4 \longrightarrow \mathbb{R}^3$ *be defined by*

$$\mathbf{T}(a, b, c, d) = (a - b + c + d, a + 2c - d, a + b + 3c - 3d), a, b, c, d \in \mathbb{R}$$

*Verify that* $\operatorname{rank}(\mathbf{T}) + \operatorname{nullity}(\mathbf{T}) = \dim(\mathcal{V}_4(\mathbb{R})$.

**Solution.** Let

$$\begin{aligned}
\mathbf{T}(1,0,0,0) &= (1,1,1) = \mathbf{v_1} \\
\mathbf{T}(0,1,0,0) &= (-1,0,1) = \mathbf{v_2} \\
\mathbf{T}(0,0,1,0) &= (1,2,3) = \mathbf{v_3} \\
\mathbf{T}(0,0,0,1) &= (1,-1,-3) = \mathbf{v_4}
\end{aligned}$$

$\mathbf{T}(\mathbb{R}^4)$ is generated by $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ and therefore a maximal independent subset of $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}$ will form a basis of $\mathbf{T}(\mathbb{R}^4)$. $\mathbf{v_1}, \mathbf{v_2}$ are linearly independent because if $\alpha\mathbf{v_1} + \beta\mathbf{v_2} = \mathbf{0}$, then $\alpha - \beta = 0, \alpha = 0$ so $\alpha = \beta = 0$.

$\mathbf{v_3}$ is dependent on $\mathbf{v_1}, \mathbf{v_2}$, because if $\mathbf{v_3} = \alpha\mathbf{v_1} + \beta\mathbf{v_2}$, then $\alpha - \beta = 1, \alpha = 2, \alpha + \beta = 3 \Rightarrow \alpha = 2, \beta = 1 \therefore \mathbf{v_3} = 2\mathbf{v_1} + \mathbf{v_2}$.

$\mathbf{v_4}$ is dependent on $\mathbf{v_1}, \mathbf{v_2}$, because if $\mathbf{v_4} = \alpha\mathbf{v_1} + \beta\mathbf{v_2}$, then $\alpha - \beta = 1, \alpha = -1, \alpha + \beta = -3 \Rightarrow \alpha = -1, \beta = -2 \therefore \mathbf{v_4} = -\mathbf{v_1} - 2\mathbf{v_2}$.

Thus $\mathbf{v_1}, \mathbf{v_2}$ is a basis of $\mathbf{T}(\mathbb{R}^4)$, so rank $\mathbf{T} = 2$.

Now $(a, b, c, d) \in \ker \mathbf{T} \Leftrightarrow a - b + c + d = 0, a + 2c - d = 0, a + b + 3c - 3d = 0$ Choosing particular values of $a, b, c, d$, we see that $(1, 2, 0, 1), (-1, 1, 1, 1) \in \ker \mathbf{T}$ and are linearly independent, so $\dim \ker \mathbf{T} \geq 2$. But $(1, 2, 0, 1), (-1, 1, 1, 1)$ generate $\ker \mathbf{T}$, because if $(a, b, c, d) \in \ker \mathbf{T}$, and $(a, b, c, d) = \alpha(1, 2, 0, 1) + \beta(-1, 1, 1, 1)$, then $\alpha - \beta = a, 2\alpha + \beta = b, \beta = c, \alpha + \beta = d$, so $\alpha = a + c, \beta = c$ and these satisfy the remaining equations $2\alpha + \beta = b, \alpha + \beta = d$, because $(a, b, c, d) \in \ker \mathbf{T}$ and therefore $a - b + c + d = 0, a + 2c - d = 0$. Thus $(a, b, c, d) = (a + c)(1, 2, 0, 1) + c(-1, 1, 1, 1)$, so $\dim \ker \mathbf{T} = \text{nullity } \mathbf{T} = 2$

Hence rank $\mathbf{T}$ + nullity $\mathbf{T} = 4 = \dim(\mathbb{R}^4)$, as required. ∎

**Question 1(c)** *If* $\mathbf{T}$ *is an operator on* $\mathbb{R}^3$ *whose basis is* $\mathbf{B} = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$ *such that*

$$[\mathbf{T} : \mathbf{B}] = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

*find a matrix of* $\mathbf{T}$ *w.r.t. a basis* $\mathbf{B_1} = \{(0, 1, -1), (1, -1, 1), (-1, 1, 0)\}$.

**Solution.** The basis $\mathbf{B}$ is the standard basis, hence the representation of $\mathbf{B_1}$ in this basis is as given. (Note that if $\mathbf{B}$ were some other basis, we would write $\mathbf{B_1}$ in that basis, and then continue as below.) Let $\mathbf{v_1} = (0, 1, -1), \mathbf{v_2} = (1, -1, 1), \mathbf{v_3} = (-1, 1, 0)$. Then

$$\begin{aligned} \mathbf{T}(\mathbf{v_1}) &= (0, 1, -1) = \mathbf{v_1} \\ \mathbf{T}(\mathbf{v_2}) &= (0, 0, 0) = \mathbf{0} \\ \mathbf{T}(\mathbf{v_3}) &= (1, -1, 0) = -\mathbf{v_3} \end{aligned}$$

Thus

$$[\mathbf{T} : \mathbf{B_1}] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

**Note:** The main idea behind the above solution is to express $\mathbf{T}(\mathbf{v_i}) = \sum_{i=0}^{n} \alpha_{ii}\mathbf{v_i}$. Now we solve for $\alpha_{ii}$ to get the matrix for $\mathbf{T}$ in the new basis.

An alternative is to compute $\mathbf{P}^{-1}[\mathbf{T} : \mathbf{B}]\mathbf{P}$, where $\mathbf{P}$ is given by $[\mathbf{v_1}, \ldots, \mathbf{v_n}] = [\mathbf{e_1}, \ldots, \mathbf{e_n}]\mathbf{P}$ or $\mathbf{P} = [\mathbf{v_1}', \ldots, \mathbf{v_n}']$. Show that this is true. ∎

**Question 2(a)** *If* $\mathbf{A} = \langle a_{ij} \rangle$ *is an* $n \times n$ *matrix such that* $a_{ii} = n, a_{ij} = r$ *if* $i \neq j$, *show that*

$$[\mathbf{A} - (n - r)\mathbf{I}][\mathbf{A} - (n - r + nr)\mathbf{I}] = \mathbf{0}$$

*Hence find the inverse of the* $n \times n$ *matrix* $\mathbf{B} = \langle b_{ij} \rangle$ *where* $b_{ii} = 1, b_{ij} = \rho, i \neq j$ *and* $\rho \neq 1, \rho \neq \frac{1}{1-n}$.

**Solution.** Let $\mathbf{C} = \mathbf{A} - (n-r)\mathbf{I}$, then every entry of $\mathbf{C}$ is $r$. Let $\mathbf{D} = \mathbf{A} - (n-r+nr)\mathbf{I} = \mathbf{C} - nr\mathbf{I}$. Thus $\mathbf{CD} = \mathbf{C}^2 - nr\mathbf{C}$. Each entry of $\mathbf{C}^2$ is $nr^2$, which is the same as each entry of $nr\mathbf{C}$, so $\mathbf{CD} = \mathbf{0}$ as required.

The given equation implies

$$\mathbf{A}[\mathbf{A} - (2n - 2r + nr)\mathbf{I}] = -(n-r)(n-r-nr)\mathbf{I}$$

Let $\mathbf{A} = n\mathbf{B}$, where $r = \rho n$. Thus $\mathbf{A}$ satisfies the conditions for the equation to hold, so substituting $\mathbf{A}$ and $r$ in the above equation

$$\begin{aligned} n\mathbf{B}[n\mathbf{B} - (2n - 2n\rho + n^2\rho)\mathbf{I}] &= -(n-n\rho)(n-n\rho-n^2\rho)\mathbf{I} \\ \mathbf{B}[\mathbf{B} - (2 - 2\rho + n\rho)\mathbf{I}] &= -(1-\rho)(1-\rho-n\rho)\mathbf{I} \\ \mathbf{B}^{-1} &= -\frac{1}{(1-\rho)(1-\rho-n\rho)}[\mathbf{B} - (2-2\rho+n\rho)\mathbf{I}] \end{aligned}$$

Thus the diagonal elements of $\mathbf{B}^{-1}$ are all $\frac{1-2\rho+n\rho}{(1-\rho)(1-\rho-n\rho)}$, while the off-diagonal elements are all $\frac{2-3\rho+n\rho}{(1-\rho)(1-\rho-n\rho)}$. ∎

**Question 2(b)** *Prove that the eigenvectors corresponding to distinct eigenvalues of a square matrix are linearly independent.*

**Solution.** Let $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_r}$ be eigenvectors corresponding to distinct eigenvalues $\lambda_1, \ldots, \lambda_r$ of a matrix $\mathbf{A}$. Let $a_1\mathbf{x_1} + \ldots + a_r\mathbf{x_r} = \mathbf{0}, a_i \in \mathbb{R}$. We shall show that $a_i = 0, 1 \leq i \leq r$.

Let $\mathbf{L_1} = \prod_{i=2}^{r}(\mathbf{A} - \lambda_i\mathbf{I})$. Note that the factors of $\mathbf{L_1}$ commute. Thus $\mathbf{L_1x_2} = \prod_{i=3}^{r}(\mathbf{A} - \lambda_i\mathbf{I})(\mathbf{A} - \lambda_2\mathbf{I})\mathbf{x_2} = \mathbf{0}$ because $\mathbf{Ax_2} = \lambda_2\mathbf{x_2}$. Similarly $\mathbf{L_1x_3} = \ldots = \mathbf{L_1x_r} = \mathbf{0}$. Moreover $\mathbf{L_1x_1} = (\lambda_1 - \lambda_2)\ldots(\lambda_1 - \lambda_r)\mathbf{x_1}$.

Consequently

$$\begin{aligned} \mathbf{0} &= \mathbf{L_1}(a_1\mathbf{x_1} + \ldots + a_r\mathbf{x_r}) \\ &= a_1\mathbf{L_1x_1} \\ &= a_1(\lambda_1 - \lambda_2)\ldots(\lambda_1 - \lambda_r)\mathbf{x_1} \end{aligned}$$

$\lambda_1 - \lambda_i \neq 0, 2 \leq i \leq r$, and $\mathbf{x_1} \neq 0$ so $a_1 = 0$.

Similarly taking $\mathbf{L_i} = \prod_{\substack{j=1 \\ i \neq j}}^{r}(\mathbf{A} - \lambda_j\mathbf{I})$, we show that $a_i = 0$ for $1 \leq i \leq r$. Thus $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_r}$ are linearly independent.

∎

**Question 2(c)** *Determine the eigenvalues and eigenvectors of the matrix*

$$\mathbf{A} = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 2 & 6 \\ 0 & 0 & 5 \end{pmatrix}$$

**Solution.** The characteristic polynomial of $\mathbf{A}$ is $|\lambda\mathbf{I} - \mathbf{A}| = (\lambda - 3)(\lambda - 2)(\lambda - 5)$.[1] Thus the eigenvalues of $\mathbf{A}$ are 3, 2, 5.

If $\mathbf{x} = (x_1, x_2, x_3)$ is an eigenvector corresponding to $\lambda = 3$ then

$$(\mathbf{A} - 3\mathbf{I})\mathbf{x} = \begin{pmatrix} 0 & 1 & 4 \\ 0 & -1 & 6 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $x_2 + 4x_3 = 0, -x_2 + 6x_3 = 0, 2x_3 = 0$, take $x_1 = 1, x_2 = 0$ to get $(1, 0, 0)$ as an eigenvector for $\lambda = 3$. All the eigenvectors are $(x_1, 0, 0), x_1 \neq 0$.

If $\mathbf{x} = (x_1, x_2, x_3)$ is an eigenvector corresponding to $\lambda = 2$ then

$$(\mathbf{A} - 2\mathbf{I})\mathbf{x} = \begin{pmatrix} 1 & 1 & 4 \\ 0 & 0 & 6 \\ 0 & 0 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $x_1 + x_2 + 4x_3 = 0, 6x_3 = 0, 3x_3 = 0$, take $x_1 = 1$ to get $(1, -1, 0)$ as an eigenvector for $\lambda = 2$. All the eigenvectors are $(x_1, -x_1, 0), x_1 \neq 0$.

If $\mathbf{x} = (x_1, x_2, x_3)$ is an eigenvector corresponding to $\lambda = 5$ then

$$(\mathbf{A} - 5\mathbf{I})\mathbf{x} = \begin{pmatrix} -2 & 1 & 4 \\ 0 & -3 & 6 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \mathbf{0}$$

Thus $-2x_1 + x_2 + 4x_3 = 0, -3x_2 + 6x_3 = 0$, take $x_3 = 1$ to get $(3, 2, 1)$ as an eigenvector for $\lambda = 5$. All eigenvectors are $(3x_3, 2x_3, x_3), x_3 \neq 0$. ∎

**Question 3(a)** *Show that the matrix congruent to a skew symmetric matrix is skew symmetric. Use the result to prove that the determinant of a skew symmetric matrix of even order is the square of a rational function of its elements.*

**Solution.** Let $\mathbf{B} = \mathbf{P}'\mathbf{A}\mathbf{P}$, be congruent to $\mathbf{A}$, and $\mathbf{A}' = -\mathbf{A}$. Then $\mathbf{B}' = \mathbf{P}'\mathbf{A}'\mathbf{P} = -\mathbf{P}'\mathbf{A}\mathbf{P} = -\mathbf{B}$, so $\mathbf{B}$ is skew symmetric.

We prove the second result by induction on $m$, where $n = 2m$ is the order of the skew symmetric matrix under consideration. If $m = 1$ then $\mathbf{A} = \left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right), |\mathbf{A}| = a^2$, so the result is true for $m = 1$.

Assume by induction that the result is true for all skew symmetric matrices of even order $< 2m$. If $\mathbf{A} \equiv \mathbf{0}$, there is nothing to prove. Otherwise there exists at least one non-zero element $a_{ij}$. Changing the first row with row $j$, we move $a_{ij}$ in the first row. Changing

[1] Note that the determinant of an upper diagonal or a lower diagonal matrix is just the product of the elements on the main diagonal.

column 1 and column $j$, we get $-a_{ij}$ in the symmetric position. Now by multiplying the new matrix by suitable elementary matrices on the left and right, we get

$$\mathbf{P}'\mathbf{A}\mathbf{P} = \begin{pmatrix} 0 & a_{ij} & * & * & \dots & * \\ -a_{ij} & 0 & * & * & \dots & * \\ * & * & & & & \\ \dots & & & & \mathbf{A_{2m-2}} & \\ * & * & & & & \end{pmatrix}$$

Now we can find $\mathbf{P}^*$ a product of elementary matrices such that

$$\mathbf{P}^{*'}\mathbf{P}'\mathbf{A}\mathbf{P}\mathbf{P}^* = \begin{pmatrix} 0 & a_{ij} & 0 & 0 & \dots & 0 \\ -a_{ij} & 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & & & & \\ \dots & & & & \mathbf{A_{2m-2}} & \\ 0 & 0 & & & & \end{pmatrix}$$

Thus $\det \mathbf{A}$ = determinant of a skew symmetric matrix of order $2 \times$ determinant of a skew symmetric matrix of order $2(m-1)$. The induction hypothesis now gives the result. ■

**Question 3(b)** *Find the rank of the matrix*

$$\mathbf{A} = \begin{pmatrix} 0 & c & -b & a' \\ -c & 0 & a & b' \\ b & -a & 0 & c' \\ -a' & -b' & -c' & 0 \end{pmatrix}$$

*where* $aa' + bb' + cc' = 0, a, b, c$ *positive integers.*

**Solution.** The rank of a skew symmetric matrix is always even[2]. Since $\begin{vmatrix} 0 & c \\ -c & 0 \end{vmatrix} = c^2 > 0, \operatorname{rank} \mathbf{A} \geq 2$.

If $c' \neq 0$, then

$$-c'|\mathbf{A}| = \begin{vmatrix} 0 & c & -b & a' \\ cc' & 0 & ac' & b'c' \\ b & -a & 0 & c' \\ -a' & -b' & -c' & 0 \end{vmatrix}$$

Adding $b'\mathbf{R_3} - a\mathbf{R_4}$ to $\mathbf{R_2}$, all the entries of the second row become 0, so $-c'|\mathbf{A}| = 0 \Rightarrow |\mathbf{A}| = 0 \Rightarrow \operatorname{rank} \mathbf{A} < 4 \Rightarrow \operatorname{rank} \mathbf{A} = 2$.

[2]We can prove this by induction on the order of the skew symmetric matrix $\mathbf{S}$. It is true if $\mathbf{S}$ is a $1 \times 1$ matrix, since it must be the 0-matrix, thus has rank 0. Now given an $(n+1) \times (n+1)$ matrix $\mathbf{S}$, we can write it as $\begin{pmatrix} \mathbf{C} & \mathbf{b} \\ -\mathbf{b}' & 0 \end{pmatrix}$, where $\mathbf{C}$ is skew symmetric, and hence has even rank by the induction hypothesis. If $\mathbf{b}$ is linearly dependent on the columns of $\mathbf{C}$, then by a series of elementary operations on $\mathbf{S}$, we can transform it into $\mathbf{P'SP} = \begin{pmatrix} \mathbf{C} & \mathbf{0} \\ \mathbf{0}' & 0 \end{pmatrix}$, so $\operatorname{rank} \mathbf{S} = \operatorname{rank} \mathbf{P'SP} = \operatorname{rank} \mathbf{C}$. If $\mathbf{b}$ is linearly independent of the columns of $\mathbf{C}$, then $\mathbf{b}$' is linearly independent of the rows of $\mathbf{C}$, so $\operatorname{rank} \mathbf{S} = \operatorname{rank}\begin{pmatrix} \mathbf{C} & \mathbf{b} \\ -\mathbf{b}' & 0 \end{pmatrix} = \operatorname{rank}[\mathbf{C}\ \mathbf{b}] + 1 (\because [-\mathbf{b}'0]$ is independent of the rows of $[\mathbf{C}\ \mathbf{b}]) = \operatorname{rank} \mathbf{C} + 2$, which is also even.

If $c' = 0$,

$$a|\mathbf{A}| = \begin{vmatrix} 0 & c & -b & a'a \\ c & 0 & a & b'a \\ b & -a & 0 & 0 \\ -a'a & -b'a & 0 & 0 \end{vmatrix}$$

Adding $-b'\mathbf{R_3}$ to $\mathbf{R_4}$, we see that the fourth row has all 0's, hence rank $\mathbf{A} = 2$ as before.

**Alternate solution:**

$$-a|\mathbf{A}| = \begin{pmatrix} 0 & c & -b & a' \\ -c & 0 & a & b' \\ b & -a & 0 & c' \\ aa' & ab' & ac' & 0 \end{pmatrix}$$

Add $-c'\mathbf{R_2}$ and $b'\mathbf{R_3}$ to $\mathbf{R_4}$, then all entries of the last row become 0. So rank $\mathbf{A} < 4$, and by the reasoning above, rank $\mathbf{A} \geq 2$, rank $\mathbf{A} \neq 3$ so rank $\mathbf{A} = 2$. ∎

**Question 3(c)** *Reduce the following symmetric matrix to a diagonal form and interpret the results in terms of quadratic forms.*

$$\mathbf{A} = \begin{pmatrix} 3 & 2 & -1 \\ 2 & 2 & 3 \\ -1 & 3 & 1 \end{pmatrix}$$

**Solution.**

$$\begin{aligned} (x\ y\ z)\mathbf{A}\left(\begin{smallmatrix} x \\ y \\ z \end{smallmatrix}\right) &= 3x^2 + 2y^2 + z^2 + 4xy - 2xz + 6yz \\ &= 3(x + \frac{2}{3}y - \frac{1}{3}z)^2 + \frac{2}{3}y^2 + \frac{2}{3}z^2 + \frac{22}{3}yz \\ &= 3(x + \frac{2}{3}y - \frac{1}{3}z)^2 + \frac{2}{3}(y + \frac{11}{2}z)^2 - \frac{117}{6}z^2 \\ &= 3X^2 + \frac{2}{3}Y^2 - \frac{117}{6}Z^2 \end{aligned}$$

where $X = x - \frac{2}{3}y - \frac{1}{3}z, Y = y + \frac{11}{2}z, Z = z$. This implies $z = Z, y = Y - \frac{11}{2}Z, x = X + \frac{2}{3}(Y - \frac{11}{2}Z) - \frac{1}{3}Z = X + \frac{2}{3}Y - 4Z$.

Then if $\mathbf{P} = \begin{pmatrix} 1 & \frac{2}{3} & -4 \\ 0 & 1 & -\frac{11}{2} \\ 0 & 0 & 1 \end{pmatrix}$, we have $\mathbf{P'AP} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & \frac{2}{3} & 0 \\ 0 & 0 & -\frac{117}{6} \end{pmatrix}$.

The quadratic form associated with $\mathbf{A}$ is indefinite as it takes both positive and negative values. Note that $\mathbf{x'Ax}$ and $\mathbf{x'P'APx'}$ take the same values. ∎