UPSC Civil Services Main 1995 - Mathematics
Linear Algebra

Brij Bhooshan

Asst. Professor

B.S.A. College of Engg & Technology

Mathura

1 Linear Algebra

Question 1(a) *Let $\mathbf{T}(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$ be a linear transformation on $\mathbb{R}^3$. What is the matrix of $\mathbf{T}$ w.r.t. the standard basis? What is a basis of the range space of $\mathbf{T}$? What is a basis of the null space of $\mathbf{T}$?*

Solution.

$$\begin{aligned}
\mathbf{T}(\mathbf{e_1}) &= \mathbf{T}(1,0,0) = (3,-2,-1) = 3\mathbf{e_1} - 2\mathbf{e_2} - \mathbf{e_3} \\
\mathbf{T}(\mathbf{e_2}) &= \mathbf{T}(0,1,0) = (0,1,2) = \mathbf{e_2} + 2\mathbf{e_3} \\
\mathbf{T}(\mathbf{e_3}) &= \mathbf{T}(0,0,1) = (1,0,4) = \mathbf{e_1} + 4\mathbf{e_3} \\
\mathbf{T} \Longleftrightarrow \mathbf{A} &= \begin{pmatrix} 3 & 0 & 1 \\ -2 & 1 & 0 \\ -1 & 2 & 4 \end{pmatrix}
\end{aligned}$$

Clearly $\mathbf{T}(\mathbf{e_2}), \mathbf{T}(\mathbf{e_3})$ are linearly independent. If $(3,-2,-1) = \alpha(0,1,2) + \beta(1,0,4)$, then $\beta = 3, \alpha = -2$, but $2\alpha + 4\beta \neq -1$, so $\mathbf{T}(\mathbf{e_1}), \mathbf{T}(\mathbf{e_2}), \mathbf{T}(\mathbf{e_3})$ are linearly independent. Thus $(3,-2,-1), (0,1,2), (1,0,4)$ is a basis of the range space of $\mathbf{T}$.

Note that $\mathbf{T}(x_1, x_2, x_3) = 0 \Leftrightarrow x_1 = x_2 = x_3 = 0$, so the null space of $\mathbf{T}$ is $\{\mathbf{0}\}$, and the empty set is a basis. Note that the matrix of $\mathbf{T}$ is nonsingular, so $\mathbf{T}(\mathbf{e_1}), \mathbf{T}(\mathbf{e_2}), \mathbf{T}(\mathbf{e_3})$ are linearly independent. ∎

Question 1(b) *Let $\mathbf{A}$ be a square matrix of order n. Prove that $\mathbf{Ax} = \mathbf{b}$ has a solution $\Leftrightarrow \mathbf{b} \in \mathbb{R}^n$ is orthogonal to all solutions $\mathbf{y}$ of the system $\mathbf{A'y} = \mathbf{0}$.*

Solution. If $\mathbf{x}$ is a solution of $\mathbf{Ax} = \mathbf{b}$ and $\mathbf{y}$ is a solution of $\mathbf{A}'\mathbf{y} = \mathbf{0}$, then $\mathbf{b}'\mathbf{y} = \mathbf{x}'\mathbf{A}'\mathbf{y} = 0$, thus $\mathbf{b}$ is orthogonal to $\mathbf{y}$.

Conversely, suppose $\mathbf{b}'\mathbf{y} = 0$ for all $\mathbf{y} \in \mathbb{R}^n$ which is a solution of $\mathbf{A}'\mathbf{y} = \mathbf{0}$. Let $\mathcal{W} = \mathbf{A}(\mathbb{R}^n) =$ the range space of $\mathbf{A}$, and $\mathcal{W}^\perp$ its orthogonal complement. If $\mathbf{A}'\mathbf{y} = \mathbf{0}$ then $\mathbf{x}'\mathbf{A}'\mathbf{y} = 0 \Rightarrow (\mathbf{Ax})'\mathbf{y} = 0$ for every $\mathbf{x} \in \mathbb{R}^n \Rightarrow \mathbf{y} \in \mathcal{W}^\perp$. Conversely $\mathbf{y} \in \mathcal{W}^\perp \Rightarrow \forall \mathbf{x} \in \mathbb{R}^n.(\mathbf{Ax})'\mathbf{y} = 0 \Rightarrow \mathbf{x}'\mathbf{A}'\mathbf{y} = 0 \Rightarrow \mathbf{A}'\mathbf{y} = \mathbf{0}$. Thus $\mathcal{W}^\perp = \{\mathbf{y} \mid \mathbf{A}'\mathbf{y} = \mathbf{0}\}$. Now $\mathbf{b}'\mathbf{y} = 0$ for all $\mathbf{y} \in \mathcal{W}^\perp$, so $\mathbf{b} \in \mathcal{W} \Rightarrow \mathbf{b} = \mathbf{Ax}$ for some $\mathbf{x} \in \mathbb{R}^n \Rightarrow \mathbf{Ax} = \mathbf{b}$ is solvable. ∎

Question 1(c) *Define a similar matrix and prove that two similar matrices have the same characteristic equation. Write down a matrix having* $1, 2, 3$ *as eigenvalues. Is such a matrix unique?*

Solution. Two matrices $\mathbf{A}, \mathbf{B}$ are said to be similar if there exists a matrix $\mathbf{P}$ such that $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$. If $\mathbf{A}, \mathbf{B}$ are similar, say $\mathbf{B} = \mathbf{P}^{-1}\mathbf{AP}$, then characteristic polynomial of $\mathbf{B}$ is $|\lambda\mathbf{I} - \mathbf{B}| = |\lambda\mathbf{I} - \mathbf{P}^{-1}\mathbf{AP}| = |\mathbf{P}^{-1}\lambda\mathbf{IP} - \mathbf{P}^{-1}\mathbf{AP}| = |\mathbf{P}^{-1}||\lambda\mathbf{I} - \mathbf{A}||\mathbf{P}| = |\lambda\mathbf{I} - \mathbf{A}|$. (Note that $|\mathbf{X}||\mathbf{Y}| = |\mathbf{XY}|$.) Thus the characteristic polynomial of $\mathbf{B}$ is the same as that of $\mathbf{A}$.

Clearly the matrix $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ has eigenvalues 1,2,3. Such a matrix is not unique, for example $\mathbf{B} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ has the same eigenvalues, but $\mathbf{B} \neq \mathbf{A}$. ∎

Question 2(a) *Show that*

$$\mathbf{A} = \begin{pmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{pmatrix}$$

is diagonalizable and hence determine $\mathbf{A}^5$.

Solution.

$$\begin{aligned} & & |\mathbf{A} - \lambda\mathbf{I}| &= 0 \\ &\Rightarrow & \begin{vmatrix} 5-\lambda & -6 & -6 \\ -1 & 4-\lambda & 2 \\ 3 & -6 & -4-\lambda \end{vmatrix} &= 0 \\ &\Rightarrow & (5-\lambda)[(4-\lambda)(-4-\lambda) + 12] + 6[4 + \lambda - 6] - 6[6 - 3(4-\lambda)] &= 0 \\ &\Rightarrow & (5-\lambda)[\lambda^2 - 4] + 6[\lambda - 2 - 3\lambda + 6] &= 0 \\ &\Rightarrow & -\lambda^3 + 5\lambda^2 + 4\lambda - 20 - 12\lambda + 24 &= 0 \\ &\Rightarrow & \lambda^3 - 5\lambda^2 + 8\lambda - 4 &= 0 \end{aligned}$$

Thus $\lambda = 1, 2, 2$.

If (x_1, x_2, x_3) is an eigenvector for $\lambda = 1$, then

$$\begin{aligned} \begin{pmatrix} 4 & -6 & -6 \\ -1 & 3 & 2 \\ 3 & -6 & -5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \mathbf{0} \\ \Rightarrow \quad 4x_1 - 6x_2 - 6x_3 &= 0 \\ -x_1 + 3x_2 + 2x_3 &= 0 \\ 3x_1 - 6x_2 - 5x_3 &= 0 \end{aligned}$$

Thus $x_1 = x_3, x_3 = -3x_2$, so $(-3, 1, -3)$ is an eigenvector for $\lambda = 1$.

If (x_1, x_2, x_3) is an eigenvector for $\lambda = 2$, then

$$\begin{aligned} \begin{pmatrix} 3 & -6 & -6 \\ -1 & 2 & 2 \\ 3 & -6 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \mathbf{0} \\ \Rightarrow \quad 3x_1 - 6x_2 - 6x_3 &= 0 \\ -x_1 + 2x_2 + 2x_3 &= 0 \\ 3x_1 - 6x_2 - 6x_3 &= 0 \end{aligned}$$

Thus $x_1 - 2x_2 - 2x_3 = 0$, so taking $x_1 = 0, x_2 = 1$, $(0, 1, -1)$ is an eigenvector for $\lambda = 2$. Taking $x_1 = 4, x_2 = 1$, $(4, 1, 1)$ is another eigenvector for $\lambda = 2$, and these two are linearly independent.

Let $\mathbf{P} = \begin{pmatrix} -3 & 0 & 4 \\ 1 & 1 & 1 \\ -3 & -1 & 1 \end{pmatrix}$. A simple calculation shows that $\mathbf{P}^{-1} = \frac{1}{2}\begin{pmatrix} 2 & -4 & -4 \\ -4 & 9 & 7 \\ 2 & -3 & -3 \end{pmatrix}$.

Clearly $\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$.

Now $\mathbf{P}^{-1}\mathbf{A}^5\mathbf{P} = (\mathbf{P}^{-1}\mathbf{A}\mathbf{P})^5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{pmatrix}$.

$$\begin{aligned} \mathbf{A}^5 &= \mathbf{P} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{pmatrix} \mathbf{P}^{-1} \\ &= \frac{1}{2} \begin{pmatrix} -3 & 0 & 4 \\ 1 & 1 & 1 \\ -3 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 32 & 0 \\ 0 & 0 & 32 \end{pmatrix} \begin{pmatrix} 2 & -4 & -4 \\ -4 & 9 & 7 \\ 2 & -3 & -3 \end{pmatrix} \\ &= \frac{1}{2} \begin{pmatrix} -3 & 0 & 128 \\ 1 & 32 & 32 \\ -3 & -32 & 32 \end{pmatrix} \begin{pmatrix} 2 & -4 & -4 \\ -4 & 9 & 7 \\ 2 & -3 & -3 \end{pmatrix} \\ &= \begin{pmatrix} 125 & -186 & -186 \\ -31 & 94 & 62 \\ 93 & -186 & -154 \end{pmatrix} \end{aligned}$$

Note: Another way of computing $\mathbf{A}^5$ is given below. This uses the characteristic polynomial of $\mathbf{A} : \mathbf{A}^3 = 5\mathbf{A}^2 - 8\mathbf{A} + 4\mathbf{I}$ and not the diagonal form, so it will *not* be permissible here.

$$
\begin{aligned}
\mathbf{A}^5 &= \mathbf{A}^2(5\mathbf{A}^2 - 8\mathbf{A} + 4\mathbf{I}) \\
&= 5\mathbf{A}(5\mathbf{A}^2 - 8\mathbf{A} + 4\mathbf{I}) - 8(5\mathbf{A}^2 - 8\mathbf{A} + 4\mathbf{I}) + 4\mathbf{A}^2 \\
&= 25(5\mathbf{A}^2 - 8\mathbf{A} + 4\mathbf{I}) - 76\mathbf{A}^2 + 84\mathbf{A} - 32\mathbf{I} \\
&= 49\mathbf{A}^2 - 116\mathbf{A} + 68\mathbf{I}
\end{aligned}
$$

Now calculate $\mathbf{A}^2$ and substitute. ∎

Question 2(b) *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be matrices of order* n*. If* $\mathbf{I} - \mathbf{AB}$ *is invertible, then* $\mathbf{I} - \mathbf{BA}$ *is also invertible and*

$$(\mathbf{I} - \mathbf{BA})^{-1} = \mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A}$$

Show that $\mathbf{AB}$ *and* $\mathbf{BA}$ *have the same characteristic values.*

Solution.

$$
\begin{aligned}
& (\mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A})(\mathbf{I} - \mathbf{BA}) \\
& = \mathbf{I} - \mathbf{BA} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A} - \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{ABA} \\
& = [\mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A}] - \mathbf{B}[\mathbf{I} + (\mathbf{I} - \mathbf{AB})^{-1}\mathbf{AB}]\mathbf{A} \qquad (1) \\
\text{Now} \quad & (\mathbf{I} - \mathbf{AB})^{-1}(\mathbf{I} - \mathbf{AB}) = (\mathbf{I} - \mathbf{AB})^{-1} - (\mathbf{I} - \mathbf{AB})^{-1}\mathbf{AB} = \mathbf{I} \\
\therefore \quad & (\mathbf{I} - \mathbf{AB})^{-1} = \mathbf{I} + (\mathbf{I} - \mathbf{AB})^{-1}\mathbf{AB} \\
\text{Substituting in (1)} \quad & (\mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A})(\mathbf{I} - \mathbf{BA}) \\
& = \mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A} - \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A} = \mathbf{I}
\end{aligned}
$$

Thus $\mathbf{I} - \mathbf{BA}$ is invertible and $(\mathbf{I} - \mathbf{BA})^{-1} = \mathbf{I} + \mathbf{B}(\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A}$ as desired.

We shall show that $\lambda\mathbf{I} - \mathbf{AB}$ is invertible if and only if $\lambda\mathbf{I} - \mathbf{BA}$ is invertible. This means that if λ is an eigenvalue of $\mathbf{AB}$, then $|\lambda\mathbf{I} - \mathbf{AB}| = 0 \Rightarrow |\lambda\mathbf{I} - \mathbf{BA}| = 0$ so λ is an eigenvalue of $\mathbf{BA}$.

If $\lambda\mathbf{I} - \mathbf{AB}$ is invertible, then

$$
\begin{aligned}
& (\mathbf{I} + \mathbf{B}(\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A})(\lambda\mathbf{I} - \mathbf{BA}) \\
& = \lambda\mathbf{I} - \mathbf{BA} + \lambda\mathbf{B}(\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A} - \mathbf{B}(\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{ABA} \\
& = \lambda[\mathbf{I} + \mathbf{B}(\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A}] - \mathbf{B}[\mathbf{I} + (\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{AB}]\mathbf{A} \qquad (2) \\
\text{Now} \quad & (\lambda\mathbf{I} - \mathbf{AB})^{-1}(\lambda\mathbf{I} - \mathbf{AB}) = \lambda(\lambda\mathbf{I} - \mathbf{AB})^{-1} - (\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{AB} = \mathbf{I} \\
\therefore \quad & \lambda(\lambda\mathbf{I} - \mathbf{AB})^{-1} = \mathbf{I} + (\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{AB} \\
\text{Substituting in (2)} \quad & (\mathbf{I} + \mathbf{B}(\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A})(\lambda\mathbf{I} - \mathbf{BA}) \\
& = \lambda\mathbf{I} + \lambda\mathbf{B}(\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A} - \lambda\mathbf{B}(\lambda\mathbf{I} - \mathbf{AB})^{-1}\mathbf{A} = \lambda\mathbf{I}
\end{aligned}
$$

Thus $\lambda\mathbf{I} - \mathbf{BA}$ is invertible if $\lambda\mathbf{I} - \mathbf{AB}$ is invertible. The converse is obvious as the situation is symmetric, thus $\mathbf{AB}$ and $\mathbf{BA}$ have the same eigenvalues.

We give another simple proof of the fact that $\mathbf{AB}$ and $\mathbf{BA}$ have the same eigenvalues.

1. Let 0 be an eigenvalue of $\mathbf{AB}$. This means that $\mathbf{AB}$ is singular, i.e. $0 = |\mathbf{AB}| = |\mathbf{A}||\mathbf{B}| = |\mathbf{BA}|$, so $\mathbf{BA}$ is singular, hence 0 is an eigenvalue of $\mathbf{BA}$.

2. Let $\lambda \neq 0$ be an eigenvalue of $\mathbf{AB}$ and let $\mathbf{x} \neq \mathbf{0}$ be an eigenvector corresponding to λ, i.e. $\mathbf{ABx} = \lambda\mathbf{x}$. Let $\mathbf{y} = \mathbf{Bx}$. Then $\mathbf{y} \neq \mathbf{0}$, because $\mathbf{Ay} = \mathbf{ABx} = \lambda\mathbf{x} \neq \mathbf{0}$ as $\lambda \neq 0$. Now $\mathbf{BAy} = \mathbf{BABx} = \mathbf{B}(\mathbf{ABx}) = \lambda\mathbf{Bx} = \lambda\mathbf{y}$. Thus λ is an eigenvalue of $\mathbf{BA}$.

∎

Question 2(c) *Let $a, b \in \mathbb{C}, |b| = 1$ and let $\mathbf{H}$ be a Hermitian matrix. Show that the eigenvalues of $a\mathbf{I} + b\mathbf{H}$ lie on a straight line in the complex plane.*

Solution. Let t be as eigenvalue of $\mathbf{H}$, which has to be real because $\mathbf{H}$ is Hermitian. Clearly $a + tb$ is an eigenvalue of $a\mathbf{I} + b\mathbf{H}$. Conversely, if λ is an eigenvalue of $a\mathbf{I} + b\mathbf{H}$, then $\frac{\lambda - a}{b}$ (note $b \neq 0$ as $|b| = 1$) is an eigenvalue of $\mathbf{H}$.

Clearly $a + tb$ lies on the straight line joining points a and $a + b$:

$$z = (1 - x)a + x(b - a), \ \ x \in \mathbb{R}$$

For the sake of completeness, we prove that the eigenvalues of a Hermitian matrix $\mathbf{H}$ are real. Let $\mathbf{z} \neq \mathbf{0}$ be an eigenvector corresponding to the eigenvalue t.

$$\begin{aligned} \mathbf{Hz} &= t\mathbf{z} \\ \Rightarrow \overline{\mathbf{z}}'\mathbf{Hz} &= t\overline{\mathbf{z}}'\mathbf{z} \\ \Rightarrow \overline{\overline{\mathbf{z}}'\mathbf{Hz}}' &= \bar{t}\overline{\mathbf{z}}'\mathbf{z} \\ \text{But } \overline{\overline{\mathbf{z}}'\mathbf{Hz}} &= \overline{\mathbf{z}}'\overline{\mathbf{H}}'\mathbf{z} = \overline{\mathbf{z}}'\mathbf{Hz} = t\overline{\mathbf{z}}'\mathbf{z} \\ \Rightarrow t\overline{\mathbf{z}}'\mathbf{z} &= \bar{t}\overline{\mathbf{z}}'\mathbf{z} \\ \Rightarrow t &= \bar{t} \quad \because \overline{\mathbf{z}}'\mathbf{z} \neq 0 \end{aligned}$$

∎

Question 3(a) *Let $\mathbf{A}$ be a symmetric matrix. Show that $\mathbf{A}$ is positive definite if and only if its eigenvalues are all positive.*

Solution. $\mathbf{A}$ is real symmetric so all eigenvalues of $\mathbf{A}$ are real. Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be eigenvalues of $\mathbf{A}$, not necessarily distinct. Let $\mathbf{x_1}$ be an eigenvector corresponding to λ_1. Since λ_1 and $\mathbf{A}$ are real, $\mathbf{x_1}$ is also real. Replacing $\mathbf{x_1}$ if necessary by $\mu\mathbf{x_1}$, μ suitable, we can assume that $||\mathbf{x_1}|| = \sqrt{\mathbf{x_1'}\mathbf{x_1}} = 1$.

Let $\mathbf{P_1}$ be an orthogonal matrix with $\mathbf{x_1}$ as its first column. Such a $\mathbf{P_1}$ exists, as will be shown at the end of this result. Clearly the first column of the matrix $\mathbf{P_1}^{-1}\mathbf{AP_1}$ is equal to $\mathbf{P_1}^{-1}\mathbf{Ax} = \lambda_1\mathbf{P_1}^{-1}\mathbf{x} = \left(\begin{smallmatrix}\lambda \\ 0 \\ 0 \\ 0\end{smallmatrix}\right)$, because $\mathbf{P_1}^{-1}\mathbf{x}$ is the first column of $\mathbf{P_1}^{-1}\mathbf{P} = \mathbf{I}$. Thus $\mathbf{P_1}^{-1}\mathbf{AP_1} = \left(\begin{smallmatrix}\lambda_1 & \mathbf{L} \\ \mathbf{0} & \mathbf{B}\end{smallmatrix}\right) = \mathbf{P_1'}\mathbf{AP_1}$ where $\mathbf{B}$ is $(n-1) \times (n-1)$ symmetric. Since $\mathbf{P_1'}\mathbf{AP_1}$ is symmetric, it follows that $\mathbf{P_1}^{-1}\mathbf{AP_1} = \mathbf{P_1'}\mathbf{AP_1} = \left(\begin{smallmatrix}\lambda_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{B}\end{smallmatrix}\right)$. Induction now gives that there exists an $(n-1) \times (n-1)$ orthogonal matrix $\mathbf{Q}$ such that $\mathbf{Q'BQ} = \begin{pmatrix}\lambda_2 & 0 & \ldots & 0 \\ \ldots & & & \\ 0 & 0 & \ldots & \lambda_n\end{pmatrix}$

where $\lambda_2, \lambda_3, \ldots, \lambda_n$ are eigenvalues of $\mathbf{B}$. Let $\mathbf{P_2} = \left(\begin{smallmatrix} 1 & \mathbf{0} \\ \mathbf{0} & \mathbf{Q} \end{smallmatrix}\right)$, then $\mathbf{P_2}$ is orthogonal and $\mathbf{P'_2 P'_1 A P_1 P_2} = \text{diagonal}[\lambda_1, \ldots, \lambda_n]$. Set $\mathbf{P} = \mathbf{P_1 P_2 \ldots P_n}$, and $(y_1, \ldots, y_n)\mathbf{P'} = \mathbf{x}$ then $\mathbf{x'Ax} = \mathbf{y'P'APy} = \sum_{i=0}^n \lambda_i^2 y_i^2$.

Since $\mathbf{P}$ is non-singular, quadratic forms $\mathbf{x'Ax}$ and $\sum_{i=0}^n \lambda_i^2 y_i^2$ assume the same values. Hence $\mathbf{A}$ is positive definite if and only if $\sum_{i=0}^n \lambda_i^2 y_i^2$ is positive definite if and only if $\lambda_i > 0$ for all i.

Result used: If $\mathbf{x_1}$ is a real vector such that $||\mathbf{x_1}|| = \sqrt{\mathbf{x'_1 x_1}} = 1$ then there exists an orthogonal matrix with $\mathbf{x_1}$ as its first column.

Proof: We have to find real column vectors $\mathbf{x_2}, \ldots, \mathbf{x_n}$ such that $||\mathbf{x_i}|| = 1, 2 \leq i \leq n$ and $\mathbf{x_2}, \ldots, \mathbf{x_n}$ is an orthonormal system i.e. $\mathbf{x'_i x_j} = 0, i \neq j$. Consider the single equation $\mathbf{x'_1 x} = 0$, where $\mathbf{x}$ is a column vector to be determined. This equation has a non-zero solution, in fact the space of solutions is of dimension $n-1$, the rank of the coefficient matrix being 1. If $\mathbf{y_2}$ is a solution, we take $\mathbf{x_2} = \frac{\mathbf{y_2}}{||\mathbf{y_2}||}$ so that $\mathbf{x'_1 x_2} = 0$.

We now consider the two equations $\mathbf{x'_1 x} = 0$, $\mathbf{x'_2 x} = 0$. Again the number of unknowns is more than the number of equations, so there is a solution, say $\mathbf{y_3}$, and take $\mathbf{x_3} = \frac{\mathbf{y_3}}{||\mathbf{y_3}||}$ to get $\mathbf{x_1}, \mathbf{x_2}, \mathbf{x_3}$ mutually orthogonal.

Proceeding in this manner, if we consider $n-1$ equations $\mathbf{x'_1 x} = 0, \ldots, \mathbf{x'_{n-1} x} = 0$, these will have a nonzero solution $\mathbf{y_n}$, so we set $\mathbf{x_n} = \frac{\mathbf{y_n}}{||\mathbf{y_n}||}$. Clearly $\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_n}$ is an orthonormal system, and therefore $\mathbf{P} = [\mathbf{x_1}, \ldots, \mathbf{x_n}]$ is an orthogonal matrix having $\mathbf{x_1}$ as a first column. ■

Question 3(b) *Let $\mathbf{A}$ and $\mathbf{B}$ be square matrices of order n, show that $\mathbf{AB} - \mathbf{BA}$ can never be equal to the identity matrix.*

Solution. Let $\mathbf{A} = \langle a_{ij} \rangle$ and $\mathbf{B} = \langle b_{ij} \rangle$. Then

$$\begin{aligned} \operatorname{tr} \mathbf{AB} &= \text{Sum of diagonal elements of } \mathbf{AB} \\ &= \sum_{i=1}^n \sum_{k=1}^n a_{ik} b_{ki} = \sum_{k=1}^n \sum_{i=1}^n b_{ki} a_{ik} = \operatorname{tr} \mathbf{BA} \end{aligned}$$

Thus $\operatorname{tr}(\mathbf{AB} - \mathbf{BA}) = \operatorname{tr} \mathbf{AB} - \operatorname{tr} \mathbf{BA} = 0$. But the trace of the identity matrix is n, thus $\mathbf{AB} - \mathbf{BA}$ can never be equal to the identity matrix. ■

Question 3(c) *Let $\mathbf{A} = \langle a_{ij} \rangle, 1 \leq i, j \leq n$. If $\sum_{\substack{j=1 \\ i \neq j}}^n |a_{ij}| < |a_{ii}|$, then the eigenvalues of $\mathbf{A}$ lie in the disc*

$$|\lambda - a_{ii}| \leq \sum_{\substack{j=1 \\ i \neq j}}^n |a_{ij}|$$

Solution. See the solution to question 2(c), year 1997. We showed that if $|\lambda - a_{ii}| > \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}|$ then $|\lambda \mathbf{I} - \mathbf{A}| \neq 0$, so λ is not an eigenvalue of $\mathbf{A}$. Thus if λ is an eigenvalue, then $|\lambda - a_{ii}| \leq \sum_{\substack{j=1 \\ i \neq j}}^{n} |a_{ij}|$, so λ lies in the disc described in the question. ∎