

UPSC Civil Services Main 1996 - Mathematics

Linear Algebra

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Mathura

Question 1(a) In \mathbb{R}^4 let \mathcal{W}_1 be the space generated by $\{(1, 1, 0, -1), (2, 4, 6, 0)\}$ and let \mathcal{W}_2 be space generated by $\{(-1, -2, -2, 2), (4, 6, 4, -6), (1, 3, 4, -3)\}$. Find a basis for the space $\mathcal{W}_1 + \mathcal{W}_2$.

Solution. Let $\mathbf{v}_1 = (1, 1, 0, -1)$, $\mathbf{v}_2 = (2, 4, 6, 0)$, $\mathbf{v}_3 = (-1, -2, -2, 2)$, $\mathbf{v}_4 = (4, 6, 4, -6)$, $\mathbf{v}_5 = (1, 3, 4, -3)$. Since $\mathbf{w} \in \mathcal{W}_1 + \mathcal{W}_2$ can be written as $\mathbf{w} = \mathbf{w}_1 + \mathbf{w}_2$, and $\mathbf{w}_1 = \alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2$ and $\mathbf{w}_2 = \alpha_3\mathbf{v}_3 + \alpha_4\mathbf{v}_4 + \alpha_5\mathbf{v}_5$, it follows that \mathbf{w} is a linear combination of $\mathbf{v}_i \Rightarrow \mathcal{W}_1 + \mathcal{W}_2$ is generated by $\{\mathbf{v}_i, 1 \leq i \leq 5\}$. Thus a maximal independent subset of $\{\mathbf{v}_i, 1 \leq i \leq 5\}$ will be a basis of $\mathcal{W}_1 + \mathcal{W}_2$.

Clearly \mathbf{v}_1 and \mathbf{v}_2 are linearly independent. If possible, let $\mathbf{v}_3 = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2$, then the four equations

$$\begin{aligned}\lambda_1 + 2\lambda_2 &= -1 \\ \lambda_1 + 4\lambda_2 &= -2 \\ 0\lambda_1 + 6\lambda_2 &= -2 \\ -\lambda_1 + 0\lambda_2 &= 2\end{aligned}$$

should be consistent and provide us λ_1, λ_2 . Clearly the third and fourth equations give us $\lambda_1 = -2, \lambda_2 = -\frac{1}{3}$ which do not satisfy the first two equations. Thus $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

If possible let $\mathbf{v}_4 = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3$. Then

$$\lambda_1 + 2\lambda_2 - \lambda_3 = 4 \tag{1a}$$

$$\lambda_1 + 4\lambda_2 - 2\lambda_3 = 6 \tag{1b}$$

$$0\lambda_1 + 6\lambda_2 - 2\lambda_3 = 4 \tag{1c}$$

$$-\lambda_1 + 0\lambda_2 + 2\lambda_3 = -6 \tag{1d}$$

Adding (1b) and (1d) we get $4\lambda_2 = 0$, so $\lambda_2 = 0$. Solving (1a) and (1b) we get $\lambda_3 = -2, \lambda_1 = 2$. These values satisfy all the four equations, so $\mathbf{v}_4 = 2\mathbf{v}_1 - 2\mathbf{v}_3$.

If possible let $\mathbf{v}_5 = \lambda_1\mathbf{v}_1 + \lambda_2\mathbf{v}_2 + \lambda_3\mathbf{v}_3$. Then

$$\lambda_1 + 2\lambda_2 - \lambda_3 = 1 \quad (2a)$$

$$\lambda_1 + 4\lambda_2 - 2\lambda_3 = 3 \quad (2b)$$

$$0\lambda_1 + 6\lambda_2 - 2\lambda_3 = 4 \quad (2c)$$

$$-\lambda_1 + 0\lambda_2 + 2\lambda_3 = -3 \quad (2d)$$

Adding (2b) and (2d) we get $4\lambda_2 = 0$, so $\lambda_2 = 0$. (2c) then gives us $\lambda_3 = -2$, and (2a) now gives $\lambda_1 = -1$, which satisfies all equations. Thus $\mathbf{v}_5 = -\mathbf{v}_1 - 2\mathbf{v}_3$. Hence $\{(1, 1, 0, -1), (2, 4, 6, 0), (-1, -2, -2, 2)\}$ is a basis of $\mathcal{W}_1 + \mathcal{W}_2$. ■

Question 1(b) Let \mathcal{V} be a finite dimensional vector space and $\mathbf{v} \in \mathcal{V}, \mathbf{v} \neq \mathbf{0}$. Show that there exists a linear functional f on \mathcal{V} such that $f(\mathbf{v}) \neq 0$.

Solution. Complete \mathbf{v} to a basis of \mathcal{V} , say $\{\mathbf{v}_1 = \mathbf{v}, \mathbf{v}_2, \dots, \mathbf{v}_n\}$, where $\dim \mathcal{V} = n$. Define $f(\mathbf{v}_j) = \delta_{1j}$ and $f(\sum_{j=1}^n a_j \mathbf{v}_j) = \sum_{j=1}^n a_j f(\mathbf{v}_j)$.

Clearly f is a linear functional over \mathcal{V} , and $f(\mathbf{v}) = f(\mathbf{v}_1) = 1$. Note that $f(\mathbf{v}_j) = 0, j > 1$ and if any $\mathbf{w} \in \mathcal{V}, \mathbf{w} = \sum_i a_i \mathbf{v}_i, f(\mathbf{w}) = a_1$. ■

Question 1(c) Let $\mathcal{V} = \mathbb{R}^3, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ be a basis of \mathcal{V} . Let $\mathbf{T} : \mathcal{V} \rightarrow \mathcal{V}$ be such that $\mathbf{T}(\mathbf{v}_i) = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, 1 \leq i \leq 3$. By writing the matrix of \mathbf{T} w.r.t. another basis show that the matrices

$$\mathbf{A} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \text{ and } \mathbf{B} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

are similar.

Solution. Clearly \mathbf{A} is the matrix of \mathbf{T} w.r.t. the basis $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. Note that

$$[\mathbf{T}(\mathbf{v}_1), \mathbf{T}(\mathbf{v}_2), \mathbf{T}(\mathbf{v}_3)] = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\mathbf{A}$$

Let

$$\begin{aligned} \mathbf{w}_1 &= \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 \\ \mathbf{w}_2 &= \mathbf{v}_1 - \mathbf{v}_2 \\ \mathbf{w}_3 &= \mathbf{v}_2 - \mathbf{v}_3 \\ \Rightarrow \mathbf{T}(\mathbf{w}_1) &= 3\mathbf{w}_1, \mathbf{T}(\mathbf{w}_2) = \mathbf{T}(\mathbf{w}_3) = 0 \end{aligned}$$

We now show that $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ is a basis for \mathcal{V} , i.e. these are linearly independent.

Let $\alpha\mathbf{w}_1 + \beta\mathbf{w}_2 + \gamma\mathbf{w}_3 = \mathbf{0}$, then $(\alpha + \beta)\mathbf{v}_1 + (\alpha - \beta + \gamma)\mathbf{v}_2 + (\alpha - \gamma)\mathbf{v}_3 = \mathbf{0}$. But $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, therefore $\alpha + \beta = 0, \alpha - \beta + \gamma = 0, \alpha - \gamma = 0 \Rightarrow \alpha = \beta = \gamma = 0 \Rightarrow \mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are linearly independent.

The matrix of \mathbf{T} w.r.t. the basis $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ is clearly \mathbf{B} . Note that the choice of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ is suggested by the shape of \mathbf{B} .

If $(\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3) = (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\mathbf{P}, |\mathbf{P}| \neq 0$ then $\mathbf{B} = \mathbf{P}^{-1}\mathbf{A}\mathbf{P}$, so \mathbf{A} and \mathbf{B} are similar. ■

Question 2(a) Let $\mathcal{V} = \mathbb{R}^3$ and $\mathbf{T} : \mathcal{V} \rightarrow \mathcal{V}$ be a linear map defined by

$$\mathbf{T}(x, y, z) = (x + z, -2x + y, -x + 2y + z)$$

What is the matrix of \mathbf{T} w.r.t. the basis $(1, 0, 1), (-1, 1, 1), (0, 1, 1)$? Using this matrix write down the matrix of \mathbf{T} with respect to the basis $(0, 1, 2), (-1, 1, 1), (0, 1, 1)$.

Solution. Let $\mathbf{v}_1 = (1, 0, 1), \mathbf{v}_2 = (-1, 1, 1), \mathbf{v}_3 = (0, 1, 1)$. $\mathbf{T}(x, y, z) = (x+z, -2x+y, -x+2y+z) = \alpha\mathbf{v}_1 + \beta\mathbf{v}_2 + \gamma\mathbf{v}_3$, say. This means $\alpha - \beta = x+z, \beta + \gamma = -2x+y, \alpha + \beta + \gamma = -x+2y+z$. This implies $\alpha = x+y+z, \beta = y, \gamma = -2x$. Thus $\mathbf{T}(x, y, z) = (x+y+z)\mathbf{v}_1 + y\mathbf{v}_2 - 2x\mathbf{v}_3$. Hence

$$[\mathbf{T}(\mathbf{v}_1) \ \mathbf{T}(\mathbf{v}_2) \ \mathbf{T}(\mathbf{v}_3)] = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ -2 & 2 & 0 \end{pmatrix}$$

Let $\mathbf{w}_1 = (0, 1, 2), \mathbf{w}_2 = (-1, 1, 1), \mathbf{w}_3 = (0, 1, 1)$. Then

$$[\mathbf{w}_1 \ \mathbf{w}_2 \ \mathbf{w}_3] = [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3] \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Hence

$$\begin{aligned} [\mathbf{T}(\mathbf{w}_1) \ \mathbf{T}(\mathbf{w}_2) \ \mathbf{T}(\mathbf{w}_3)] &= [\mathbf{T}(\mathbf{v}_1) \ \mathbf{T}(\mathbf{v}_2) \ \mathbf{T}(\mathbf{v}_3)]\mathbf{P} \\ &= [\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]\mathbf{A}\mathbf{P} \\ &= [\mathbf{w}_1 \ \mathbf{w}_2 \ \mathbf{w}_3]\mathbf{P}^{-1}\mathbf{A}\mathbf{P} \end{aligned}$$

where

$$\mathbf{A} = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ -2 & 2 & 0 \end{pmatrix}, \quad \mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus the matrix of \mathbf{T} w.r.t. basis $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ is

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 1 \\ -2 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 2 \\ -2 & 0 & -1 \\ 0 & 2 & 0 \end{pmatrix}$$

■

Question 2(b) Let \mathcal{V} and \mathcal{W} be finite dimensional vector spaces such that $\dim \mathcal{V} \geq \dim \mathcal{W}$. Show that there is always a linear map of \mathcal{V} onto \mathcal{W} .

Solution. Let $\mathbf{w}_1, \mathbf{w}_2, \dots, \mathbf{w}_m$ be a basis of \mathcal{W} , and $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ be a basis of \mathcal{V} , $n \geq m$. Define

$$\begin{aligned}\mathbf{T}(\mathbf{v}_i) &= \mathbf{w}_i, \quad i = 1, 2, \dots, m \\ \mathbf{T}(\mathbf{v}_i) &= \mathbf{0}, \quad i = m + 1, \dots, n\end{aligned}$$

and for any $\mathbf{v} \in \mathcal{V}$, $\mathbf{v} = \sum_{i=1}^n \alpha_i \mathbf{v}_i$, $\mathbf{T}(\mathbf{v}) = \sum_{i=1}^m \alpha_i \mathbf{T}(\mathbf{v}_i)$.

Clearly $\mathbf{T} : \mathcal{V} \rightarrow \mathcal{W}$ is linear. \mathbf{T} is onto, since if $\mathbf{w} \in \mathcal{W}$, $\mathbf{w} = \sum_{i=1}^m a_i \mathbf{w}_i$, then $\mathbf{T}(\sum_{i=1}^m a_i \mathbf{v}_i) = \sum_{i=1}^m a_i \mathbf{T}(\mathbf{v}_i) = \mathbf{w}$, proving the result. ■

Question 2(c) Solve by Cramer's rule

$$\begin{aligned}x + y - 2z &= 1 \\ 2x - 7z &= 3 \\ x + y - z &= 5\end{aligned}$$

Solution.

$$D = \begin{vmatrix} 1 & 1 & -2 \\ 2 & 0 & -7 \\ 1 & 1 & -1 \end{vmatrix} = \begin{vmatrix} -1 & -1 & -2 \\ -5 & -7 & -7 \\ 0 & 0 & -1 \end{vmatrix} = -2$$

$$x = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 3 & 0 & -7 \\ 5 & 1 & -1 \end{vmatrix}}{D} = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 3 & 0 & -7 \\ 4 & 0 & 1 \end{vmatrix}}{D} = \frac{-31}{-2} = \frac{31}{2}$$

$$y = \frac{\begin{vmatrix} 1 & 1 & -2 \\ 2 & 3 & -7 \\ 1 & 5 & -1 \end{vmatrix}}{D} = \frac{\begin{vmatrix} 1 & 0 & 0 \\ 2 & 1 & -3 \\ 1 & 4 & 1 \end{vmatrix}}{D} = \frac{13}{-2} = -\frac{13}{2}$$

$$z = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 1 & 1 & 5 \end{vmatrix}}{D} = \frac{\begin{vmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \\ 0 & 0 & 4 \end{vmatrix}}{D} = \frac{-8}{-2} = 4$$

Question 3(a) Find the inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

by computing its characteristic polynomial.

Solution. The characteristic polynomial of \mathbf{A} is

$$\begin{aligned} |\mathbf{A} - \lambda\mathbf{I}| &= \begin{vmatrix} -\lambda & 1 & 0 & 0 \\ 0 & -\lambda & 1 & 0 \\ 0 & 0 & -\lambda & 1 \\ 1 & 0 & 0 & -\lambda \end{vmatrix} \\ &= -\lambda[-\lambda^3] - 1[1] = \lambda^4 - 1 = 0 \end{aligned}$$

Thus by the Cayley-Hamilton theorem, $\mathbf{A}^4 = \mathbf{I}$, so $\mathbf{A}^{-1} = \mathbf{A}^3$.

$$\begin{aligned} \mathbf{A}^2 &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \\ \mathbf{A}^3 &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} = \mathbf{A}^{-1} \end{aligned}$$

■

Question 3(b) If \mathbf{A} and \mathbf{B} are $n \times n$ matrices such that $\mathbf{AB} = \mathbf{BA}$, show that \mathbf{AB} and \mathbf{BA} have a common characteristic vector.

Solution. Let λ be any eigenvalue of \mathbf{A} and let \mathcal{V}_λ be the eigenspace of \mathbf{A} corresponding to λ . We show that $\mathbf{B}(\mathcal{V}_\lambda) \subseteq \mathcal{V}_\lambda$. Let $\mathbf{v} \in \mathcal{V}_\lambda$, then $\mathbf{A}(\mathbf{B}\mathbf{v}) = \mathbf{B}(\mathbf{A}\mathbf{v}) = \mathbf{B}(\lambda\mathbf{v}) = \lambda\mathbf{B}\mathbf{v} \Rightarrow \mathbf{B}\mathbf{v} \in \mathcal{V}_\lambda$.

Consider $\mathbf{B}^* : \mathcal{V}_\lambda \rightarrow \mathcal{V}_\lambda$ such that $\mathbf{B}^*(\mathbf{v}) = \mathbf{B}(\mathbf{v})$ — note that \mathbf{B}^* is a restriction of \mathbf{B} to \mathcal{V}_λ and we have already shown that $\mathbf{B}(\mathcal{V}_\lambda) \subseteq \mathcal{V}_\lambda$.

Let μ be an eigenvalue of \mathbf{B}^* , then μ is also an eigenvalue of \mathbf{B} (because a basis of \mathcal{V}_λ can be extended to a basis of \mathcal{V} , and in this basis $\mathbf{B} = \begin{pmatrix} \mathbf{B}^* & \mathbf{C} \\ \mathbf{0} & \mathbf{D} \end{pmatrix}$ for some matrices \mathbf{C}, \mathbf{D}). Let $\mathbf{v} \in \mathcal{V}_\lambda$ be an eigenvector of \mathbf{B}^* corresponding to μ , by definition $\mathbf{v} \neq \mathbf{0}$. Then $\mathbf{B}\mathbf{v} = \mathbf{B}^*\mathbf{v} = \mu\mathbf{v}$. Thus \mathbf{A} and \mathbf{B} have a common eigenvector \mathbf{v} , note that $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ as $\mathbf{v} \in \mathcal{V}_\lambda$. ■

Question 3(c) Reduce to canonical form the orthogonal matrix

$$\mathbf{O} = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{3}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{3}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix}$$

Solution. Before solving this particular problem, we present a general discussion about orthogonal matrices. An orthogonal matrix satisfies $\mathbf{O}'\mathbf{O} = \mathbf{I}$, so its determinant is 1 or -1, here we focus on the case where $|\mathbf{O}| = 1$. If λ is an eigenvalue of \mathbf{O} and \mathbf{x} a corresponding eigenvector, then $|\lambda|^2\mathbf{x}'\mathbf{x} = (\mathbf{O}\mathbf{x})'\mathbf{O}\mathbf{x} = \mathbf{x}'\mathbf{O}'\mathbf{O}\mathbf{x} = \mathbf{x}'\mathbf{x}$, so $|\lambda| = 1$. Since the characteristic

polynomial has real coefficients, the eigenvalues must be real or in complex conjugate pairs. Thus for a matrix of order 3, at least one eigenvalue is real, and must be 1 or -1. Since $|\mathbf{O}| = 1$, one real value must be 1, and the three possibilities are $\{1, 1, 1\}$, $\{1, -1, -1\}$ and $\{1, e^{i\theta}, e^{-i\theta}\}$.

Here we consider the third case, as the given matrix has 1 and $\frac{1}{3} \pm i\frac{2\sqrt{2}}{3}$ as eigenvalues, proved later.

Let $\mathbf{Z} = \mathbf{X}_1 + i\mathbf{X}_2$ be an eigenvector corresponding to the eigenvalue $e^{i\theta}$. Let \mathbf{X}_3 be the eigenvector corresponding to the eigenvalue 1. Since \mathbf{Z} and \mathbf{X}_3 correspond to different eigenvalues, these are orthogonal, i.e. $\mathbf{Z}'\mathbf{X}_3 = (\mathbf{X}_1' + i\mathbf{X}_2')\mathbf{X}_3 = 0 \Rightarrow \mathbf{X}_1'\mathbf{X}_3 = 0, \mathbf{X}_2'\mathbf{X}_3 = 0$. Note that $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are real vectors. Since $\mathbf{O}\mathbf{Z} = e^{i\theta}\mathbf{Z} = (\cos\theta + i\sin\theta)(\mathbf{X}_1 + i\mathbf{X}_2)$. Equating real and imaginary parts we get

$$\begin{aligned} \mathbf{O}\mathbf{X}_1 &= \mathbf{X}_1 \cos\theta - \mathbf{X}_2 \sin\theta \\ \mathbf{O}\mathbf{X}_2 &= \mathbf{X}_1 \sin\theta + \mathbf{X}_2 \cos\theta \\ \therefore \mathbf{X}_1'\mathbf{O}'\mathbf{O}\mathbf{X}_1 &= (\mathbf{X}_1' \cos\theta - \mathbf{X}_2' \sin\theta)(\mathbf{X}_1 \cos\theta - \mathbf{X}_2 \sin\theta) \\ \Rightarrow \mathbf{X}_1'\mathbf{X}_1 &= \mathbf{X}_1'\mathbf{X}_1 \cos^2\theta - \mathbf{X}_2'\mathbf{X}_1 \cos\theta \sin\theta - \mathbf{X}_1'\mathbf{X}_2 \sin\theta \cos\theta + \mathbf{X}_2'\mathbf{X}_2 \sin^2\theta \\ \Rightarrow 0 &= \mathbf{X}_1'\mathbf{X}_1 \sin^2\theta - \mathbf{X}_2'\mathbf{X}_2 \sin^2\theta + 2\mathbf{X}_1'\mathbf{X}_2 \cos\theta \sin\theta \\ \Rightarrow 0 &= \mathbf{X}_1'\mathbf{X}_1 \sin\theta - \mathbf{X}_2'\mathbf{X}_2 \sin\theta + 2\mathbf{X}_1'\mathbf{X}_2 \cos\theta \quad (1) \end{aligned}$$

(Note that $\sin\theta \neq 0$ since we are considering the case where $e^{i\theta}$ is complex.) Similarly

$$\begin{aligned} \mathbf{X}_2'\mathbf{O}'\mathbf{O}\mathbf{X}_1 &= (\mathbf{X}_1' \sin\theta + \mathbf{X}_2' \cos\theta)(\mathbf{X}_1 \cos\theta - \mathbf{X}_2 \sin\theta) \\ \Rightarrow \mathbf{X}_2'\mathbf{X}_1 &= \mathbf{X}_1'\mathbf{X}_1 \sin\theta \cos\theta - \mathbf{X}_1'\mathbf{X}_2 \sin^2\theta - \mathbf{X}_2'\mathbf{X}_2 \sin\theta \cos\theta + \mathbf{X}_2'\mathbf{X}_1 \cos^2\theta \\ \Rightarrow 0 &= \mathbf{X}_1'\mathbf{X}_1 \cos\theta - \mathbf{X}_1'\mathbf{X}_2 \cos\theta - 2\mathbf{X}_1'\mathbf{X}_2 \sin\theta \quad (2) \end{aligned}$$

Multiplying (1) by $\sin\theta$ and (2) by $\cos\theta$ and adding, we get $\mathbf{X}_1'\mathbf{X}_1 - \mathbf{X}_2'\mathbf{X}_2 = 0$ or $\mathbf{X}_1'\mathbf{X}_1 = \mathbf{X}_2'\mathbf{X}_2$, so from (2), $\mathbf{X}_1'\mathbf{X}_2 = 0$, i.e. $\mathbf{X}_1, \mathbf{X}_2$ are orthogonal.

Thus $\mathbf{X}_1, \mathbf{X}_2, \mathbf{X}_3$ are mutually orthogonal. We can assume that $\mathbf{X}_1'\mathbf{X}_1 = \mathbf{X}_2'\mathbf{X}_2 = 1$, replacing \mathbf{Z} by $\lambda\mathbf{Z}$, $\lambda \in \mathbb{R}$ if necessary. Similarly we can take $\mathbf{X}_3'\mathbf{X}_3 = 1$. Let $\mathbf{P} = [\mathbf{X}_1 \ \mathbf{X}_2 \ \mathbf{X}_3]$ so that $\mathbf{P}'\mathbf{P} = \mathbf{I}$. Now

$$\begin{aligned} \mathbf{O}[\mathbf{X}_1 \ \mathbf{X}_2 \ \mathbf{X}_3] &= [\mathbf{X}_1 \cos\theta - \mathbf{X}_2 \sin\theta, \mathbf{X}_1 \sin\theta + \mathbf{X}_2 \cos\theta, \mathbf{X}_3] \\ &= [\mathbf{X}_1 \ \mathbf{X}_2 \ \mathbf{X}_3] \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ \Rightarrow \mathbf{P}^{-1}\mathbf{O}\mathbf{P} = \mathbf{P}'\mathbf{O}\mathbf{P} &= \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

which is the canonical form of \mathbf{O} when the eigenvalues are $1, e^{i\theta}, e^{-i\theta}$.

Solution of given problem.

$$\begin{aligned}
 \mathbf{O} &= \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \end{pmatrix} \\
 |\mathbf{O} - \lambda \mathbf{I}| &= \begin{vmatrix} \frac{2}{3} - \lambda & -\frac{2}{3} & \frac{1}{3} \\ \frac{2}{3} & \frac{1}{3} - \lambda & -\frac{2}{3} \\ \frac{1}{3} & \frac{2}{3} & \frac{2}{3} - \lambda \end{vmatrix} = \frac{1}{27} \begin{vmatrix} 2 - 3\lambda & -2 & 1 \\ 2 & 1 - 3\lambda & -2 \\ 1 & 2 & 2 - 3\lambda \end{vmatrix} \\
 &= \frac{1}{27} [(2 - 3\lambda)^2(1 - 3\lambda) + 4(2 - 3\lambda) + 1(3 + 3\lambda) + 2(6 - 6\lambda)] \\
 &= -\frac{1}{27} [27\lambda^3 - 45\lambda^2 + 45\lambda - 27] \\
 &= -\frac{1}{3} [(\lambda - 1)(3\lambda^2 - 2\lambda - 3)]
 \end{aligned}$$

Thus $\lambda = 1, \frac{1}{3} \pm i\frac{2\sqrt{2}}{3}$ are eigenvalues of \mathbf{O} .

Thus the canonical form of \mathbf{O} is derived from above, where $\cos \theta = \frac{1}{3}, \sin \theta = \frac{2\sqrt{2}}{3}$:

$$\begin{pmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The matrix \mathbf{P} can be determined as follows (this is not needed for this problem, but is given for completeness):

1. Let (x_1, x_2, x_3) be an eigenvector for $\lambda = 1$, then

$$\begin{aligned}
 -\frac{1}{3}x_1 - \frac{2}{3}x_2 + \frac{1}{3}x_3 &= 0 \\
 \frac{2}{3}x_1 - \frac{2}{3}x_2 - \frac{2}{3}x_3 &= 0 \\
 \frac{1}{3}x_1 + \frac{2}{3}x_2 - \frac{1}{3}x_3 &= 0
 \end{aligned}$$

Thus $x_2 = 0, x_1 - x_3 = 0$, so we can take $(1, 0, 1)$ as an eigenvector.

2. The vectors $\mathbf{X}_1, \mathbf{X}_2$ in the above discussion are determined by the requirements

$$\begin{aligned}
 \mathbf{O}\mathbf{X}_1 &= \mathbf{X}_1 \cos \theta - \mathbf{X}_2 \sin \theta \\
 \mathbf{O}\mathbf{X}_2 &= \mathbf{X}_1 \sin \theta + \mathbf{X}_2 \cos \theta
 \end{aligned}$$

where $\cos \theta = \frac{1}{3}$, $\sin \theta = \frac{2\sqrt{2}}{3}$. This gives us the following equations

$$2x_{11} - 2x_{12} + x_{13} = x_{11} - x_{21}2\sqrt{2} \quad (3)$$

$$2x_{11} + x_{12} - 2x_{13} = x_{12} - x_{22}2\sqrt{2} \quad (4)$$

$$x_{11} + 2x_{12} + 2x_{13} = x_{13} - x_{23}2\sqrt{2} \quad (5)$$

$$2x_{21} - 2x_{22} + x_{23} = x_{11}2\sqrt{2} + x_{21} \quad (6)$$

$$2x_{21} + x_{22} - 2x_{23} = x_{12}2\sqrt{2} + x_{22} \quad (7)$$

$$x_{21} + 2x_{22} + 2x_{23} = x_{13}2\sqrt{2} + x_{23} \quad (8)$$

Adding the last 3 equations, we get $\sqrt{2}x_{21} = x_{11} + x_{12} + x_{13}$. Subtracting equation (6) from (8), $\sqrt{2}x_{22} = x_{13} - x_{11}$, and from (7) $\sqrt{2}x_{23} = x_{11} - x_{12} + x_{13}$. Substituting these in the first 3 equations and simplifying, we get $x_{11} = -x_{13}$. Setting $x_{11} = 0, x_{12} = 1$, we get $(0, 1, 0), (\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$ as a possible solution for $\mathbf{X}_1, \mathbf{X}_2$.

Putting these together we get

$$\mathbf{P} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & 1 \\ 1 & 0 & 0 \\ 0 & -\frac{1}{\sqrt{2}} & 1 \end{pmatrix}$$

We can now verify that $\mathbf{OP} = \mathbf{P} \begin{pmatrix} \frac{1}{3} & \frac{2\sqrt{2}}{3} & 0 \\ -\frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ■