# UPSC Civil Services Main 1998 - Mathematics
# Linear Algebra

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**Question 1(a)** *Given two linearly independent vectors* $(1, 0, 1, 0)$ *and* $(0, -1, 1, 0)$ *of* $\mathbb{R}^4$*, find a basis of* $\mathbb{R}^4$ *which includes them.*

**Solution.** Let $\mathbf{v_1} = (1, 0, 1, 0), \mathbf{v_2} = (0, -1, 1, 0)$. Clearly these are linearly independent. Let $\mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}$ be the standard basis. Then $\mathbf{v_1}, \mathbf{v_2}, \mathbf{e_1}, \mathbf{e_2}, \mathbf{e_3}, \mathbf{e_4}$ generate $\mathbb{R}^4$. We have to find four vectors out of these which are linearly independent and include $\mathbf{v_1}, \mathbf{v_2}$.

If $\alpha\mathbf{v_1} + \beta\mathbf{v_2} + \gamma\mathbf{e_1} = 0$, then $\alpha + \gamma = 0, -\alpha = 0, \alpha + \beta = 0 \Rightarrow \alpha = \beta = \gamma = 0$. Therefore $\mathbf{v_1}, \mathbf{v_2}, \mathbf{e_1}$ are linearly independent.

We now show that $\mathbf{v_1}, \mathbf{v_2}, \mathbf{e_1}, \mathbf{e_4}$ are linearly independent. Let $\alpha\mathbf{v_1} + \beta\mathbf{v_2} + \gamma\mathbf{e_1} + \delta\mathbf{e_4} = 0$ then $\delta = 0$, and therefore $\alpha = \beta = \gamma = 0$ because $\mathbf{v_1}, \mathbf{v_2}, \mathbf{e_1}$ are linearly independent.

Thus $\mathbf{v_1}, \mathbf{v_2}, \mathbf{e_1}, \mathbf{e_4}$ is a basis of $\mathbb{R}^4$.

Note that $\mathbf{e_2} = \mathbf{v_1} - \mathbf{v_2} - \mathbf{e_1}, \mathbf{e_3} = \mathbf{v_1} - \mathbf{e_1}$. ∎

**Question 1(b)** *If* $\mathcal{V}$ *is a finite dimensional vector space over* $\mathbb{R}$ *and if* $f$ *and* $g$ *are two linear transformations from* $\mathcal{V}$ *to* $\mathbb{R}$ *such that* $f(\mathbf{v}) = 0$ *implies* $g(\mathbf{v}) = 0$*, then prove that* $g = \lambda f$ *for some* $\lambda \in \mathbb{R}$*.*

**Solution.** If $g = 0$, take $\lambda = 0$, so $g(\mathbf{v}) = 0 = 0f(\mathbf{v})$ for all $\mathbf{v} \in \mathcal{V}$.

If $g \neq 0$, then $f \neq 0$. Thus $\exists \mathbf{v} \in \mathcal{V}$ such that $f(\mathbf{v}) \neq 0 \Rightarrow \exists \mathbf{w} \in \mathcal{V}$ such that $f(\mathbf{w}) = 1$ (Note that $f(\frac{\mathbf{v}}{f(\mathbf{v})}) = 1$).

Thus $\mathcal{V}/\ker f \backsimeq \mathbb{R}$, or $\dim(\ker f) = n - 1$. Similarly $\ker g$ has dimension $n - 1$. In fact, $\ker f = \ker g \because \ker f \subseteq \ker g$ and $\dim(\ker f) = \dim(\ker g)$. Let $\{\mathbf{v_2}, \ldots, \mathbf{v_n}\}$ be a basis of $\ker f$ and extend it to $\{\mathbf{v_1}, \mathbf{v_2}, \ldots, \mathbf{v_n}\}$ a basis of $\mathcal{V}$. Then $g = \lambda f$ with $\lambda = g(\mathbf{v_1})/f(\mathbf{v_1}) \because$ if $\mathbf{v} = \alpha_1\mathbf{v_1} + \ldots + \alpha_n\mathbf{v_n}$, then $g(\mathbf{v}) = \alpha_1 g(\mathbf{v_1}) = \alpha_1\lambda f(\mathbf{v_1}) = \lambda f(\mathbf{v})$. ∎

**Question 1(c)** *Let $\mathbf{T} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be defined by $\mathbf{T}(x_1, x_2, x_3) = (x_2, x_3, -cx_1 - bx_2 - ax_3)$ where $a, b, c$ are fixed real numbers. Show that $\mathbf{T}$ is a linear transformation of $\mathbb{R}^3$ and that $\mathbf{A}^3 + a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = \mathbf{0}$ where $\mathbf{A}$ is the matrix of $\mathbf{T}$ w.r.t. the standard basis of $\mathbb{R}^3$.*

**Solution.** Let $\mathbf{x} = (x_1, x_2, x_3), \mathbf{y} = (y_1, y_2, y_3)$. Then

$$\begin{aligned}
\mathbf{T}(\alpha\mathbf{x} + \beta\mathbf{y}) &= (\alpha x_2 + \beta y_2, \alpha x_3 + \beta x_3, -c(\alpha x_1 + \beta y_1) - b(\alpha x_2 + \beta y_2) - a(\alpha x_3 + \beta y_3)) \\
&= \alpha(x_2, x_3, -cx_1 - bx_2 - ax_3) + \beta(y_2, y_3, -cy_1 - by_2 - ay_3) \\
&= \alpha\mathbf{T}(\mathbf{x}) + \beta\mathbf{T}(\mathbf{y})
\end{aligned}$$

Thus $\mathbf{T}$ is linear.

Clearly

$$\begin{aligned}
\mathbf{T}(1, 0, 0) &= (0, 0, -c) \\
\mathbf{T}(0, 1, 0) &= (1, 0, -b) \\
\mathbf{T}(0, 0, 1) &= (0, 1, -a)
\end{aligned}$$

$$\mathbf{A} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -c & -b & -a \end{pmatrix}$$

The characteristic equation of $\mathbf{A}$ is $|\mathbf{A} - \lambda\mathbf{I}| = 0$.

$$\begin{aligned}
\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & \lambda & 1 \\ -c & -b & -a - \lambda \end{vmatrix} &= 0 \\
-\lambda^2(a + \lambda) - b\lambda - c &= 0 \\
\lambda^3 + a\lambda^2 + b\lambda + c &= 0
\end{aligned}$$

Now by the Cayley-Hamilton theorem $\mathbf{A}^3 + a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I} = \mathbf{0}$. ∎

**Question 2(a)** *If $\mathbf{A}$ and $\mathbf{B}$ are two matrices of order $2 \times 2$ such that $\mathbf{A}$ is skew-Hermitian and $\mathbf{AB} = \mathbf{B}$ then show that $\mathbf{B} = \mathbf{0}$.*

**Solution.** We first of all prove that eigenvalues of skew-Hermitian matrices are 0 or pure imaginary. Let $\mathbf{A}$ be skew-Hermitian, i.e. $\overline{\mathbf{A}}' = -\mathbf{A}$ and let $\lambda$ be its characteristic root. If $\mathbf{x}$ is an eigenvector of $\lambda$, then

$$\begin{aligned}
\mathbf{Ax} &= \lambda\mathbf{x} \\
\Rightarrow \quad \overline{\mathbf{x}}'\lambda\mathbf{x} &= \overline{\mathbf{x}}'\mathbf{Ax} \\
&= -\overline{\mathbf{x}}'\overline{\mathbf{A}}'\mathbf{x} \\
&= -\overline{\mathbf{Ax}}'\mathbf{x} \\
&= -\overline{\lambda}\overline{\mathbf{x}}'\mathbf{x}
\end{aligned}$$

Thus $\lambda = -\overline{\lambda} \quad \because \overline{\mathbf{x}}'\mathbf{x} \neq 0$, showing that the real part of $\lambda$ is 0.

Now if $\mathbf{B} \neq \mathbf{0}$ and $\mathbf{c_1}, \mathbf{c_2}$ are the columns of $\mathbf{B}$, then $\mathbf{c_1} \neq \mathbf{0}$ or $\mathbf{c_2} \neq \mathbf{0}$. $\mathbf{AB} = \mathbf{B}$ means that $\mathbf{Ac_1} = \mathbf{c_1}$ and $\mathbf{Ac_2} = \mathbf{c_2}$. Since either $\mathbf{c_1} \neq \mathbf{0}$ or $\mathbf{c_2} \neq \mathbf{0}$, 1 must be an eigenvalue of $\mathbf{A}$, which is not possible. Hence $\mathbf{c_1} = \mathbf{0}$ and $\mathbf{c_2} = \mathbf{0}$, which means $\mathbf{B} = \mathbf{0}$. ∎

**Question 2(b)** *If* $\mathbf{T}$ *is a complex matrix of order* $2 \times 2$ *such that* $\operatorname{tr}\mathbf{T} = \operatorname{tr}\mathbf{T}^2 = 0$*, then show that* $\mathbf{T}^2 = \mathbf{0}$*.*

**Solution.** Let $\lambda_1, \lambda_2$ be the eigenvalues of $\mathbf{T}$, then $\lambda_1^2, \lambda_2^2$ are the eigenvalues of $\mathbf{T}^2$. Given that

$$\begin{aligned} \operatorname{tr}\mathbf{T} &= \lambda_1 + \lambda_2 = 0 \\ \operatorname{tr}\mathbf{T}^2 &= \lambda_1^2 + \lambda_2^2 = 0 \end{aligned}$$

$0 = \lambda_1^2 + \lambda_2^2 = \lambda_1^2 + (-\lambda_1)^2 \Rightarrow \lambda_1 = 0$ and from $\lambda_1 + \lambda_2 = 0$ we get $\lambda_1 = \lambda_2 = 0$. The characteristic equation of $\mathbf{T}$ is $(x - \lambda_1)(x - \lambda_2) = 0$, or $x^2 = 0$. By Cayley-Hamilton theorem, we immediately get $\mathbf{T}^2 = \mathbf{0}$. ∎

**Question 2(c)** *Prove that a necessary and sufficient condition for an* $n \times n$ *real matrix* $\mathbf{A}$ *to be similar to a diagonal matrix is that the set of characteristic vectors of* $\mathbf{A}$ *includes a set of* $n$ *linearly independent vectors.*

**Solution.**

**Necessity:** By hypothesis there exists a nonsingular matrix $\mathbf{P}$ such that

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

Let $\mathbf{P} = [\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}]$, where each $\mathbf{c_i}$ is an $n$-row column vector.

$$\mathbf{A}[\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}] = [\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}]\mathbf{D} = [\lambda_1\mathbf{c_1}, \lambda_2\mathbf{c_2}, \dots, \lambda_n\mathbf{c_n}]$$

so $\mathbf{A}\mathbf{c_i} = \lambda_i\mathbf{c_i}$ for $i = 1, \dots, n$. Thus $\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}$ are characteristic vectors of $\mathbf{A}$ corresponding to the eigenvalues $\lambda_1, \dots, \lambda_n$. Since $\mathbf{P}$ is nonsingular, $\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}$ are linearly independent. Thus the set of characteristic vectors of $\mathbf{A}$ includes a set of $n$ linearly independent vectors.

**Sufficiency:** Let $\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}$ be $n$ linearly independent eigenvectors of $\mathbf{A}$ corresponding to eigenvalues $\lambda_1, \dots, \lambda_n$. Thus $\mathbf{A}\mathbf{c_i} = \lambda_i\mathbf{c_i}$ for $i = 1, \dots, n$. Let $\mathbf{P} = [\mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}]$, then $\mathbf{P}$ is nonsingular (otherwise 0 is an eigenvalue of $\mathbf{P}$, so $\exists \mathbf{x} = (x_1, \dots, x_n) \neq \mathbf{0}$ such that $\mathbf{P}\mathbf{x} = \mathbf{0} \Rightarrow x_1\mathbf{c_1} + \dots + x_n\mathbf{c_n} = \mathbf{0} \Rightarrow \mathbf{c_1}, \mathbf{c_2}, \dots, \mathbf{c_n}$ are not linearly independent.). Clearly

$$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \mathbf{D} = \begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$$

∎

**Question 3(a)** *Let* $\mathbf{A}$ *be a* $m \times n$ *matrix. Show that the sum of the rank and nullity of* $\mathbf{A}$ *is* $n$.

**Solution.** The matrix $\mathbf{A}$ can be regarded as a linear transformation $\mathbf{A} : \mathcal{F}^n \longrightarrow \mathcal{F}^m$ where $\mathcal{F}$ is the field to which the entries of $\mathbf{A}$ belong, and the bases for $\mathcal{F}^n, \mathcal{F}^m$ are standard bases.

Let $\mathbf{T} : \mathcal{V} \longrightarrow \mathcal{W}$ be a linear transformation, where $\dim(\mathcal{V}) = n, \dim(\mathcal{W}) = m$. We shall show that $\dim(\mathbf{T}(\mathcal{V})) + \dim(\operatorname{kernel} \mathbf{T}) = n$.

Take $\mathbf{v_{n-r+1}}, \ldots, \mathbf{v_n}$ to be any basis of $\operatorname{kernel} \mathbf{T}$, where $\dim(\operatorname{kernel} \mathbf{T}) = r$. Complete it to a basis $\mathbf{v_1}, \ldots, \mathbf{v_{n-r+1}} \ldots, \mathbf{v_n}$ of $\mathcal{V}$. We shall show that $\mathbf{T}(\mathbf{v_1}), \ldots, \mathbf{T}(\mathbf{v_{n-r}})$ are linearly independent and generate $\mathbf{T}(\mathcal{V})$, thus $\dim(\mathbf{T}(\mathcal{V})) = n - r$.

If $\mathbf{w} \in \mathbf{T}(\mathcal{V})$, then $\exists \mathbf{v} \in \mathcal{V}$ such that $\mathbf{T}(\mathbf{v}) = \mathbf{w}$. If $\mathbf{v} = \alpha_1 \mathbf{v_1} + \ldots + \alpha_n \mathbf{v_n}, \alpha_i \in \mathcal{F}$, then $\mathbf{w} = \mathbf{T}(\mathbf{v}) = \alpha_1 \mathbf{T}(\mathbf{v_1}) + \ldots + \alpha_{n-r} \mathbf{T}(\mathbf{v_{n-r}})$ because $\mathbf{T}(\mathbf{v_i}) = \mathbf{0}$ for $i > n - r$. Thus $\mathbf{T}(\mathcal{V})$ is generated by $\mathbf{T}(\mathbf{v_1}), \ldots, \mathbf{T}(\mathbf{v_{n-r}})$.

If $\alpha_1 \mathbf{T}(\mathbf{v_1}) + \ldots + \alpha_{n-r} \mathbf{T}(\mathbf{v_{n-r}}) = \mathbf{0}$, then $\mathbf{T}(\alpha_1 \mathbf{v_1} + \ldots + \alpha_{n-r} \mathbf{v_{n-r}}) = \mathbf{0}$. This implies $\alpha_1 \mathbf{v_1} + \ldots + \alpha_{n-r} \mathbf{v_{n-r}} \in \operatorname{kernel} \mathbf{T} \Rightarrow \alpha_1 \mathbf{v_1} + \ldots + \alpha_{n-r} \mathbf{v_{n-r}} = \alpha_{n-r+1} \mathbf{v_{n-r+1}} + \ldots + \alpha_n \mathbf{v_n}$. But $\mathbf{v_1}, \ldots, \mathbf{v_n}$ are linearly independent, so $\alpha_i = 0$ for $i = 1, \ldots, n$. Hence $\mathbf{T}(\mathbf{v_1}), \ldots, \mathbf{T}(\mathbf{v_{n-r}})$ are linearly independent, so they form a basis for $\mathbf{T}(\mathcal{V})$. Thus $\dim(\mathbf{T}(\mathcal{V})) + \dim(\operatorname{kernel} \mathbf{T}) = n$. ∎

**Question 3(b)** *Find all real* $2 \times 2$ *matrices* $\mathbf{A}$ *with real eigenvalues which satisfy* $\mathbf{A}\mathbf{A}' = \mathbf{I}$.

**Solution.** Since $\mathbf{A}\mathbf{A}' = \mathbf{I}$, $|\mathbf{A}| = \pm 1$. If $|\mathbf{A}| = 1$, then

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

so $a^2 + b^2 = 1, c^2 + d^2 = 1, ac + bd = 0, ad - bc = 1$. Let $a = \cos\theta, b = \sin\theta$. Then

$$\begin{matrix} c\cos\theta + d\sin\theta = 0 \\ -c\sin\theta + d\cos\theta = 1 \end{matrix} \Rightarrow \begin{matrix} c\cos\theta\sin\theta + d\sin^2\theta = 0 \\ -c\sin\theta\cos\theta + d\cos^2\theta = \cos\theta \end{matrix} \Rightarrow d = \cos\theta, c = -\sin\theta$$

Thus $\mathbf{A} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$, $\theta$ is real.

Now the eigenvalues of $\mathbf{A}$ are given by

$$|\mathbf{A} - \lambda\mathbf{I}| = \begin{vmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{vmatrix} = 0$$

So $(\cos\theta - \lambda)^2 + \sin^2\theta = 0$, or $\lambda^2 - 2\lambda\cos\theta + 1 = 0$. Thus

$$\lambda = \frac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm i\sin\theta$$

Since the eigenvalues of $\mathbf{A}$ are real, $\sin\theta = 0$, so $\cos\theta = \pm 1$. Thus

$$\mathbf{A} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

If $|\mathbf{A}| = -1$, $\mathbf{J} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, then $|\mathbf{JA}| = 1$. Also $\mathbf{JA}(\mathbf{JA})' = \mathbf{JAA}'\mathbf{J}' = \mathbf{JJ}' = \mathbf{I}$. Thus

$$\mathbf{JA} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$\mathbf{A} = \mathbf{J}^{-1}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix}$$

Now the eigenvalues of $\mathbf{A}$ are given by

$$0 = |\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda + \sin\theta & -\cos\theta \\ -\cos\theta & \lambda - \sin\theta \end{vmatrix} = \lambda^2 - \sin^2\theta - \cos^2\theta = \lambda^2 - 1$$

Hence $\lambda = \pm 1$, so the eigenvalues are always real. Thus the possible values of $\mathbf{A}$ are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} -\sin\theta & \cos\theta \\ \cos\theta & \sin\theta \end{pmatrix} \text{ for all real } \theta$$

∎

**Question 3(c)** *Reduce to diagonal matrix by rational congruent transformation the symmetric matrix*

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ -1 & 3 & 1 \end{pmatrix}$$

**Solution.** The corresponding quadratic form is

$$\begin{aligned} & x^2 + z^2 + 4xy - 2xz + 6yz \\ = \; & (x + 2y - z)^2 - 4y^2 + 10yz \\ = \; & (x + 2y - z)^2 - 4(y - \frac{5}{4}z)^2 + \frac{25}{4}z^2 \\ = \; & X^2 - 4Y^2 + \frac{25}{4}Z^2 \end{aligned}$$

where $X = x + 2y - z, Y = y - 5z/4, Z = z$. From this we get $z = Z, y = Y + 5Z/4, x = X - 2Y - \frac{3}{2}Z$. Thus

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & \frac{25}{4} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -\frac{3}{2} & \frac{5}{4} & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & -1 \\ 2 & 0 & 3 \\ -1 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2 & -\frac{3}{2} \\ 0 & 1 & \frac{5}{4} \\ 0 & 0 & 1 \end{pmatrix}$$

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