UPSC Civil Services Main 2001 - Mathematics Linear Algebra

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Mathura

Question 1(a) *Show that the vectors* $(1, 0, -1), (0, -3, 2)$ *and* $(1, 2, 1)$ *form a basis of the vector space* $\mathbb{R}^3(\mathbb{R})$.

Solution. Since $\dim_{\mathbb{R}}(\mathbb{R}^3) = 3$, it is enough to prove that these are linearly independent. If possible, let

$$a(1, 0, -1) + b(0, -3, 2) + c(1, 2, 1) = 0$$

This implies

$$a + c = 0, -3b + 2c = 0, -a + 2b + c = 0$$

Solving for c, $c + \frac{4}{3}c + c = 0$, so $c = 0$, hence $a = b = 0$. (Note that if these linearly independent vectors were not a basis, they could be completed into one, but in $\mathbb{R}^3$ any four vectors are linearly dependent, so this is a maximal linearly independent set, hence it is a basis.

Alternate Solution. Since $\dim(\mathbb{R}^3) = 3$, to show that $(1, 0, -1), (0, -3, 2)$ and $(1, 2, 1)$ form a basis it is enough to show that these vectors generate $\mathbb{R}^3$. In fact, given (x_1, x_2, x_3), we can always find a, b, c s.t. $(x_1, x_2, x_3) = a(1, 0, -1) + b(0, -3, 2) + c(1, 2, 1)$ as follows: $a + c = x_1, -3b + 2c = x_2, -a + 2b + c = x_3$. Thus $(c - x_1) + 2(2c - x_2)/3 + c = x_3$, so $c + \frac{4}{3}c + c = x_1 + \frac{2}{3}x_2 + x_3$. Thus $c = \frac{3x_1+2x_2+3x_3}{10}$, $a = x_1 - c = \frac{7x_1-2x_2-3x_3}{10}$, and $b = \frac{2c-x_2}{3} = \frac{x_1-x_2+x_3}{5}$. ∎

Question 1(b) *If* λ *is a characteristic root of a non-singular matrix* $\mathbf{A}$*, then prove that* $\frac{|\mathbf{A}|}{\lambda}$ *is a characteristic root of* **Adj A**.

Solution. If μ is a characteristic root of $\mathbf{A}$, then $a\mu$ is a characteristic root of $a\mathbf{A}$ for a constant a, because if $\mathbf{Av} = \mu\mathbf{v}$, $\mathbf{v} \neq 0$ a vector, then $a\mathbf{Av} = a\mu\mathbf{v}$. Hence the result.

If λ is the characteristic root of $\mathbf{A}$, $|\mathbf{A}| \neq 0$, then $\lambda \neq 0$, and λ^{-1} is a characteristic root of $\mathbf{A}^{-1}$, because $\mathbf{Av} = \lambda\mathbf{v} \Longrightarrow \mathbf{A}^{-1}\mathbf{v} = \lambda^{-1}\mathbf{v}$.

Since $\mathbf{Adj\ A} = \mathbf{A}^{-1}|\mathbf{A}|$, it follows that $\frac{|\mathbf{A}|}{\lambda}$ is a characteristic root of **Adj A**. ∎

Question 2(a) *If* $\mathbf{A} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ *show that for all integers* $n \geq 3, \mathbf{A}^n = \mathbf{A}^{n-2} + \mathbf{A}^2 - \mathbf{I}$. *Hence determine* $\mathbf{A}^{50}$.

Solution. Characteristic equation of $\mathbf{A}$ is

$$\begin{vmatrix} \lambda - 1 & 0 & 0 \\ 1 & \lambda & 1 \\ 0 & 1 & \lambda \end{vmatrix} = 0$$

or $(\lambda - 1)(\lambda^2 - 1) = \lambda^3 - \lambda^2 - \lambda + 1 = 0$. From the Cayley-Hamilton theorem, $\mathbf{A}^3 - \mathbf{A}^2 - \mathbf{A} + \mathbf{I} = 0 \Rightarrow \mathbf{A}^3 = \mathbf{A} + \mathbf{A}^2 - \mathbf{I}$. Thus the result is true for $n = 3$. Suppose the theorem is true for $n = m$ i.e. $\mathbf{A}^m = \mathbf{A}^{m-2} + \mathbf{A}^2 - \mathbf{I}$. We shall prove it for $m + 1$.

$$\begin{aligned} \mathbf{A}^{m+1} &= \mathbf{A}^m \mathbf{A} \\ &= (\mathbf{A}^{m-2} + \mathbf{A}^2 - \mathbf{I})\mathbf{A} \\ &= \mathbf{A}^{m-1} + \mathbf{A}^3 - \mathbf{A} \\ &= \mathbf{A}^{m-1} + \mathbf{A}^2 + \mathbf{A} - \mathbf{A} - \mathbf{I} \\ &= \mathbf{A}^{m-1} + \mathbf{A}^2 - \mathbf{I} \end{aligned}$$

The result follows by induction.

Let $n = 2m$. Using successively $\mathbf{A}^n = \mathbf{A}^{n-2} + \mathbf{A}^2 - \mathbf{I}$, we get $\mathbf{A}^{2m} = m\mathbf{A}^2 - (m-1)\mathbf{I}$. Now

$$\mathbf{A}^2 = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix}$$

so

$$\begin{aligned} \mathbf{A}^{50} &= 25\mathbf{A}^2 - 24\mathbf{I} \\ &= \begin{pmatrix} 25 & 0 & 0 \\ 25 & 25 & 0 \\ 25 & 0 & 25 \end{pmatrix} - \begin{pmatrix} 24 & 0 & 0 \\ 0 & 24 & 0 \\ 0 & 0 & 24 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 25 & 1 & 0 \\ 25 & 0 & 1 \end{pmatrix} \end{aligned}$$

∎

Question 2(b) *When is a square matrix* $\mathbf{A}$ *said to be congruent to a square matrix* $\mathbf{B}$? *Prove that every matrix congruent to a skew-symmetric matrix is skew-symmetric.*

Solution. $\mathbf{A} \equiv \mathbf{B}$ if $\exists \mathbf{P}$ nonsingular, s.t. $\mathbf{P}'\mathbf{A}\mathbf{P} = \mathbf{B}$. If $\mathbf{S}' = -\mathbf{S}$ then $(\mathbf{P}'\mathbf{S}\mathbf{P})' = \mathbf{P}'\mathbf{S}'\mathbf{P} = -(\mathbf{P}'\mathbf{S}\mathbf{P})$, so $\mathbf{P}'\mathbf{S}\mathbf{P}$ is also skew-symmetric. ∎

Question 2(c) *Determine the orthogonal matrix* $\mathbf{P}$ *such that* $\mathbf{P}^{-1}\mathbf{AP}$ *is diagonal where* $\mathbf{A} = \begin{pmatrix} 7 & 4 & -4 \\ 4 & -8 & -1 \\ -4 & -1 & -8 \end{pmatrix}$.

Solution. The characteristic equation is

$$\begin{aligned} \begin{vmatrix} \lambda-7 & -4 & 4 \\ -4 & \lambda+8 & 1 \\ 4 & 1 & \lambda+8 \end{vmatrix} &= 0 \\ (\lambda-7)((\lambda+8)^2-1)+4(-4-4\lambda-32)+4(-4-4\lambda-32) &= 0 \\ \lambda^3+9\lambda^2-81\lambda-729 &= 0 \\ (\lambda+9)(\lambda^2-81) &= 0 \end{aligned}$$

Thus $\lambda = 9, -9, -9$.

1. $\lambda = 9$. If (x_1, x_2, x_3) is the eigenvector corresponding to $\lambda = 9$, we get

$$\begin{aligned} 2x_1 - 4x_2 + 4x_3 &= 0 \\ -4x_1 + 17x_2 + x_3 &= 0 \\ 4x_1 + x_2 + 17x_3 &= 0 \end{aligned}$$

   From the second and third we get $18x_2 + 18x_3 = 0$. Take $x_2 = 1$. Then $x_3 = -1, x_1 = 4$, so $(4, 1, -1)$ is an eigenvector for $\lambda = 9$.

2. $\lambda = -9$. If (x_1, x_2, x_3) is the eigenvector corresponding to $\lambda = -9$, we get

$$\begin{aligned} -16x_1 - 4x_2 + 4x_3 &= 0 \\ -4x_1 - x_2 + x_3 &= 0 \\ 4x_1 + x_2 - x_3 &= 0 \end{aligned}$$

   There is only one equation $4x_1 + x_2 - x_3 = 0$. Take $x_1 = 0, x_2 = 1$, then $x_3 = 1$, so $(0, 1, 1)$ is an eigenvector. Take $x_1 = -1, x_2 = 2$, then $x_3 = -2$, so $(-1, 2, -2)$ is another eigenvector. These two are orthogonal to each other and are eigenvectors for $\lambda = -9$. Note that to make the second vector orthogonal to the first, we needed to ensure $x_2 = -x_3$, then the equation suggested values for x_1, x_2.

Let

$$\mathbf{P} = \begin{pmatrix} 0 & -\frac{1}{3} & \frac{4}{\sqrt{18}} \\ \frac{1}{\sqrt{2}} & \frac{2}{3} & \frac{1}{\sqrt{18}} \\ \frac{1}{\sqrt{2}} & -\frac{2}{3} & -\frac{1}{\sqrt{18}} \end{pmatrix}$$

Clearly $\mathbf{P}'\mathbf{P} = \mathbf{I}$, since the columns of $\mathbf{P}$ are mutually orthogonal unit vectors. Moreover from $\mathbf{A}\mathbf{x} = \mathbf{x}\lambda$ for the eigenvalues and eigenvectors, it follows that $\mathbf{AP} = \mathbf{P}\begin{pmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$.

Thus $\mathbf{P}^{-1}\mathbf{AP} = \begin{pmatrix} -9 & 0 & 0 \\ 0 & -9 & 0 \\ 0 & 0 & 9 \end{pmatrix}$, which is diagonal as required. ∎

Question 2(d) *Show that the real quadratic form*

$$\Phi = n(x_1^2 + x_2^2 + \ldots + x_n^2) - (x_1 + x_2 + \ldots + x_n)^2$$

in n variables is positive semi-definite.

Solution. Consider the expression

$$\begin{aligned} E &= (X - x_1)^2 + \ldots + (X - x_n)^2 \\ &= nX^2 - 2X(x_1 + \ldots + x_n) + (x_1^2 + x_2^2 + \ldots + x_n^2) \end{aligned}$$

Clearly E being the sum of squares is non-negative, i.e. $E \geq 0$. Let

$$A = \frac{(x_1 + x_2 + \ldots + x_n)}{n} \qquad B = \frac{(x_1^2 + x_2^2 + \ldots + x_n^2)}{n}$$

Then $E = n(X^2 - 2AX + B) = n((X - A)^2 + B - A^2)$. When $X = A$, $E = n(B - A^2) = \Phi$, and since $E \geq 0$, $\Phi \geq 0$.

If $x_1 = x_2 = \ldots = x_n = 1$, then $\Phi = 0$ showing that Φ is actually positive semi-definite.

Alternate solution. By Cauchy's inequality

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \geq \left(\sum_{i=1}^{n} a_i b_i\right)^2$$

Setting $b_1 = b_2 = \ldots = b_n = 1$, we get

$$n\left(\sum_{i=1}^{n} a_i^2\right) - \left(\sum_{i=1}^{n} a_i\right)^2 \geq 0$$

showing that Φ is positive semi-definite. ∎