MATHEMATICS

Time Allowed: 3 hours ***Maximum Marks: 300***

Candidates should attempt any FIVE questions.
All questions carry equal marks.

PAPER-I

1. (a) Let T be the linear operator in R^3 defined by
$T(x_1, x_2, x_3) = (3x_1 + x_3, -2x_1 + x_2, -x_1 + 2x_2 + 4x_3)$.
What is the matrix of T in the standard ordered basis for R^3? What is a basis of range space of T and a basis of null space of T ?

(b) Let A be a square matrix of order n. Prove that AX = b has a solution if and only if $b \in R^n$ is orthogonal to all solutions Y of the system $A^TY = 0$.

(c) Define a similar matrix. Prove that the characteristic equation of two similar matrices is the same. Let 1, 2, 3 be the eigen-values of a matrix. Write down such a matrix. Is such a matrix unique?

2. (a) Show that $A = \begin{bmatrix} 5 & -6 & -6 \\ -1 & 4 & 2 \\ 3 & -6 & -4 \end{bmatrix}$ is diagonalizable and hence determine A^5.

(b) Let A and B be matrices of order n. Prove that if (I – AB) is invertible, then (I-BA) is also invertible and $(I-BA)^{-1} = I + B(I-AB)^{-1} A$.
Show that AB and BA have precisely the same characteristic values.

(c) If a and b are complex numbers such that |b| = 1 and H is a Hermitian matrix, show that the eigen-values of aI + bH lie on a straight line in the complex plane.

3. (a) Let A be a symmetric matrix. Show that A is positive definite if and only if its eigen-values are all positive.

(b) Let A and B be square matrices of order n. Show that AB-BA can never be equal to unit matrix.

(c) Let $A = [a_{ij}]$; i, j = 1, 2, …, n and
$|a_{ii}| > \sum_{j \neq i} |a_{ij}|$ for every i = 1, 2, …. n. Show that A is a non-singular matrix. Hence or otherwise prove that the eigen-values of A lie in the discs
$$|\lambda - a_{ii}| \le \sum_{j \neq i} |a_{ij}| \; j = 1, 2, \ldots, n$$
in the complex plane.

4. (a) If g is the inverse of f and
$$f'(x) = \frac{1}{1+x^3},$$
prove that $g'(x) = 1 + [g(x)]^3$.

(b) Taking the nth derivative of $(x^n)^2$ in two different ways, show that
$$1 + \frac{n^2}{1^2} + \frac{n^2(n-1)^2}{1^2.2^2} + \frac{n^2(n-1)^2(n-2)^2}{1^2.2^2.3^2} + \ldots \text{ to } (n+1) \text{ terms} = \frac{(2n)!}{(n!)^2}$$

(c) Let f(x,y), which possesses continuous partial derivatives of second order, be a homogeneous function of x and y of degree n. Prove that

$$x^2f_{xx} + 2xyf_{xy} + y^2f_{yy} = n(n-1)f$$

5. (a) Find the area bounded by the curve

$$\left(\frac{x^2}{4}+\frac{y^2}{9}\right)^2 = \frac{x^2}{4}-\frac{y^2}{9}$$

(b) Let f(x), $x \geq 1$ be such that the area bounded by the curve y = f(x) and the lines x = 1, x=b is equal to $\sqrt{1+b^2}-\sqrt{2}$ for all $b \geq 1$. Does f attain its minimum ? If so what is its value?

(c) Show that

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right).....\Gamma\left(\frac{n-1}{n}\right) = \frac{(2\pi)}{\sqrt{n}}\frac{n-1}{2}$$

6. (a) Two conjugate semi-diameters of the ellipse

$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ cut the circle $x^2+y^2=r^2$ at P and Q.

Show that the locus of the middle point of PQ is

$$a^2\left\{\left(x^2+y^2\right)^2 - r^2x^2\right\} + b^2\left\{\left(x^2+y^2\right)^2 - r^2y^2\right\} = 0$$

(b) If the normal at one of the extremities of latus rectum of the conic $\frac{1}{r}=1+e\cos\theta$, meets the curve again at Q, show that

$$SQ = \frac{l\left(1+3e^2+e^4\right)}{\left(1+e^2-e^4\right)},$$

where S is the focus of the conic.

(c) Through a point P(x', y', z') a plane is drawn at right angles to OP to meet the coordinate axes in A, B, C. Prove that the area of the triangle ABC is $\frac{r^5}{2x'y'z'}$, where r is the measure of OP.

7. (a) Two spheres of radii r_1 and r_2 cut orthogonally. Prove that the area of the common circle is

$$\frac{\pi r_1^2 r_2^2}{r_1^2+r_2^2}$$

(b) Show that a plane through one member of the λ-system and one member of μ-system is tangent plane to the hyperboloid at the point of intersection of the two generators.

(c) Prove that the parallels through the origin to the binormals of the helix

$$x=a\cos\theta,\ y=a\sin\theta,\ z = k\theta$$

lie upon the right cone $a^2(x^2+y^2) = k^2z^2$.

8. (a) Determine a family of curves for which the ratio of the y-intercept of the tangent to the radius vector is a constant.

(b) Solve

$$(2x^2+3y^2-7)\,x\,dx - (3x^2+2y^2-8)\,y\,dy = 0$$

(c) Test whether the equation

$(x+y)^2\,dx - (y^2-2xy-x^2)\,dy=0$

is exact and hence solve it.

9. (a) Solve

$$x^3\frac{d^3y}{dx^3}+2x^2\frac{d^2y}{dx^2}+2y=10\left(x+\frac{1}{x}\right)$$

(b) Determine all real valued solutions of the equation

$$y'''-iy''+y'-iy = 0, \; y' \equiv \frac{dy}{dx}$$

(c) Find the solution of the equation
$y'' + 4y = 8 \cos 2x$,
give that $y = 0$ and $y' = 2$ when $x = 0$.

10. (a) Consider a physical entity that is specified by twenty-seven numbers A_{ijk} in a given coordinate system. In the transition to another coordinate system of this kind, let $A_{ijk}\, B_{jk}$ transform as a vector for any choice of the anti-symmetric tensor B_{jk}. Prove that the quantities $A_{ijk} - A_{ikj}$ are the components of a tensor of third order. Is A_{ijk} the components of a tensor? Give reasons for your answer.

(b) Let the region V be bounded by the smooth surface S and let n denote outward drawn unit normal vector at a point on S. If ϕ is harmonic in V, show that

$$\int_S \frac{\partial \phi}{\partial n} dS = 0.$$

(c) In the vector field $\upsilon(x)$, let there exist a surface on which $\upsilon=0$. Show that, at an arbitrary point of this surface, curl υ is tangential to the surface or vanishes.

11. (a) Prove that for the common catenary the radius of curvature at any point of the curve is equal to the length of the normal intercepted between the curve and the directrix.

(b) Two uniform rods AB and AC, smoothly jointed at A, are in equilibrium in a vertical plane. The ends B and C rest on a smooth horizontal plane and the middle points of AB and AC are connected by a string. Show that the tension of the string is $\frac{W}{(\tan B + \tan C)}$, where W is the total weight of the rods and B and C are the inclinations to the horizontal of the rods AB and AC.

(c) A semi-ellipse bounded by its minor axis is just immersed in a liquid the density of which varies as the depth. If the minor axis be in the surface, find the eccentricity in order that the focus may be the centre of pressure.

12. (a) Two bodies, of masses M and M', are attached to the lower end of an elastic string whose upper end is fixed and hang at rest; M' falls off. Show that the distance of M from the upper end of the string at time t is

$$a + b + c\cos\left(\sqrt{\frac{g}{b}}t\right),$$

where a is the unstretched length of the string, and b and c the distances by which it would be stretched when supporting M and M' respectively.

(b) A particle of mass m moves under a central attractive force $m\mu\left(\frac{5}{r^3} + \frac{8c^2}{r^5}\right)$ and is projected from an apse at a distance c with velocity $\frac{3\sqrt{\mu}}{c}$. Prove that the orbit is

$$r = c\cos\left(\frac{2\theta}{3}\right),$$

and that it will arrive at the origin after a time

$$\frac{\pi}{8}\frac{c^2}{\sqrt{\mu}}.$$

(c) If t be the time in which a projectile reaches a point P in its path and t' the time from P till it reaches the horizontal plane through the point of projection, show that the height of P above the horizontal plane is $\frac{1}{2}gtt'$ and the maximum height is $\frac{1}{8}g\left(t+t'\right)^2$.

MATHEMATICS

Time Allowed: 3 hours **Maximum Marks: 300**

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SECTION - A

1. (a) Let G be a finite set closed under an associative binary operation such that
$ab = ac$ implies $b = c$
and $ba = ca$ implies $b = c$
for all $a, b, c \in G$. Prove that G is a group.

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(b) Let G be a group of order p^n, where p is a prime number and $n > 0$. Let H be a proper subgroup of G and $N(H) = \{x \in G : x^{-1} h x \in H \ \forall\ h \in H\}$. Prove that $N(H) \neq H$.

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(c) Show that a group of order 112 is not simple.

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2. (a) Let R be a ring with identity. Suppose there is an element a of R which has more than one right inverse. Prove that a has infinitely many right inverses.

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(b) Let F be a field and let p(x) be an irreducible polynomial over F. Let $< p(x) >$ be the ideal generated by p(x). Prove that $< p(x) >$ is a maximal ideal.

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(c) Let F be a field of characteristic $p \neq 0$. Let F(x) be the polynomial ring. Suppose
$f(x) = a_0 + a_1x + \ldots + a_nx^n$
is an element of F[x]. Define
$f'(x) = a_1 + 2a_2x + \ldots + n\,a_nx^{n-1}$
If $f'(x) = 0$, prove that there exists $g(x) \in F[x]$ such that $f(x) = g(x^p)$.

3. (a) Let K and F be nonempty disjoint closed subsets of $\mathbb{R}^2$. If K is bounded, show that there exists $\delta > 0$ such that $d(x, y) \geq \delta$ for $x \in K$ and $y \in F$' where d(x,y) is the usual distance between x and y.

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(b) Let f be a continuous real function on $\mathbb{R}$ such that f maps open interval onto open intervals. Prove that f is monotonic.

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(c) Let $c_n \geq 0$ for all positive integers n such that is convergent. Suppose $\{s_n\}$ is a sequence of distinct points in (a, b). For $x \in [a,b]$, define
$$\alpha(x) = \sum_{\{n : x > s_n\}} c_n.$$
Prove that α is an increasing function. If f a continuous real function on [a, b], show that
$$\int_a^b f d\alpha = \sum_n c_n f(s_n)$$

4. (a) Suppose f maps an open ball $U \subset \mathbb{R}^n$ into $\mathbb{R}^m$ and f is differentiable on U. Suppose there exists a real number $M > 0$ such that $\| f'(x) \| \leq M$ for all $x \in U$. Prove that

$$|f(b) - f(a)| \leq M\,|b-a|$$

for all $a, b \in U$.

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(b) Find and classify the extreme values of the function

$$f(x,y) = x^2 + y^2 + x + y + xy$$

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(c) Suppose α is a real number not equal to $n\pi$ for any integer n. Prove that

$$\int_0^\infty \int_0^\infty e^{-\left(x^2+2xy\cos\alpha+y^2\right)}dxdy = \frac{\alpha}{2\sin\alpha}$$

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5. (a) Let $u(x, y) = 3x^2y + 2x^2 - y^3 - 2y^2$. Prove that u is a harmonic function. Find a harmonic function v such that u + iv is an analytic function of z.

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(b) Find the Taylor series expansion of the function $f(z) = \frac{z}{z^4+9}$ around z = 0. Find also the radius of convergence of the obtained series.

(c) Let C be the circle |z|=2 described counter clockwise. Evaluate the integral

$$\int_C dz \frac{\cosh \pi z}{z\left(z^2+1\right)}$$

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6. (a) Let $a \geq 0$. Evaluate the integral

$$\int_0^\infty \frac{\cos ax}{x^2+1}dx$$

with the aid of residues.

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(b) Let f be analytic in the entire complex plane. Suppose that there exists a constant A> 0 such that $| f(z) | \leq A\,| z |$ for all z. Prove that there exists a complex number a such that f(z) = az for all z.

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(c) Suppose a power series $\sum_{n=0}^{\infty} a_n z^n$ converges at a point $z_0 \neq 0$. Let z_1 be such that $|z_1| < |z_0|$ and $z_1 \neq 0$. Show that the series converges uniformly in the disc $\{z : |z| \leq |z_1|\}$.

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7. (a) In the context of a partial differential equation of the first order in three independent variables, define and illustrate the terms:

(i) the complete integral

(ii) the singular integral

(b) Find the general integral of

$$(y+z+w)\frac{\partial w}{\partial x}+(z+x+w)\frac{\partial w}{\partial y}+(x+y+w)\frac{\partial w}{\partial z}=x+y+z$$

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(c) Obtain the differential equation of the surfaces which are the envelopes of a one-parameter family of planes.

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8. (a) Explain in detail the Charpit's method of solving the nonlinear partial differential equation

$$f\ x, y, z, \left(\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}\right) = 0$$

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(b) Solve

$$\frac{\partial z}{\partial x_1} \frac{\partial z}{\partial x_2} \frac{\partial z}{\partial x_3} = z^3 x_1 x_2 x_3$$

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(c) Solve

$(D_x^3 - 7D_x D_y^2 - 6D_y^3)\ z = \sin(x + 2y) + e^{3x+y}$

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SECTION - B

9. (a) How do you characterize

(i) the simplest dynamical system ?

(ii) the most general dynamical system?

Show that the equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial q_k} - \frac{\partial T}{\partial q_k} = Q_k, (k = 1, 2, \ldots, n)$$

correspond to a non-conservative but sceleronomic and holonomic system with n degrees of freedom, where q, q_k, Q_k are respectively the generalized coordinates, the generalized velocities and the generalized forces.

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(b) A solid uniform sphere has a light rod rigidly attached to it which passes through the centre. The rod is so joined to a fixed vertical axis that the angle θ between the rod and the axis may alter but the rod must turn with the axis. If the vertical axis be forced to revolve constantly with uniform angular velocity, show that θ^2 is of the form

$n^2 (\cos\theta - \cos\beta)(\cos\alpha - \cos\theta)$

where n, α, β, are certain constants.

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(c) A uniform rod of length 20 cms which has one end attached to a fixed point by a light inextensible string of length $4\frac{1}{6}$ cms, is performing small oscillations in a vertical plane about its position of equilibrium. Find its position at any time and the periods of principal oscillations.

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10. (a) A carriage is placed on an inclined plane making an angle a with the horizon and rolls down without slipping between the wheels and the plane. The floor of the carriage is parallel to the plane and a perfectly rough ball is placed freely on it. Show that the acceleration of the carriage down the plane is

$$\frac{14M + 4m' + 14m}{14M + 4m' + 21m} g \sin\alpha$$

where M is the mass of the carriage excluding the wheels, m the sum of the masses of the wheels which are uniform circular discs and M' that of the ball which is a homogeneous solid sphere (the friction between the wheels and the axes is neglected). Show that for the motion to be possible, the coefficient of friction between the wheels and the plane must exceed the constant

$$\frac{7(M + m) + 2M'}{14M + 21m + 4M'} \tan\alpha$$

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(b) A sphere of radius a is projected up an inclined plane with velocity V and angular velocity Ω in the sense which would cause it to roll up; if V> a Ω and the coefficient of friction greater than $\frac{2}{7}\tan\alpha$, show that the sphere will cease to ascend at the end of a time

$$\frac{5V+2a\Omega}{5g\sin\alpha}$$

where α is the inclination of the plane.

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11. (a) Determine the restrictions on f_1, f_2, f_3 if

$$f_1(t)\frac{x^2}{a^2}+f_2(t)\frac{y^2}{b^2}+f_3(t)\frac{z^2}{c^2}=1$$

is a possible boundary surface of a liquid.

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(b) If a, b, c, d, e, f are arbitrary constants, what type of fluid motion does the velocity

(a + by – cz, d – bx + ez, f + cx – ey)

represent ?

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(c) If the fluid fill the region of spaces on the positive side of x-axis, which is a rigid boundary and if there be a source +m at the point (0, a) and an equal sink at (0, b) and if the pressure on the negative side of the boundary be the same as the pressure of the fluid at infinity, show that the resultant pressure on the boundary is

$$\pi\rho m^2\frac{(a-b)^2}{ab(a+b)}$$

where ρ is the density of the fluid.

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12. (a) Find the positive root of $\log_e x = \cos x$ nearest to five places of decimal by Newton-Raphson method.

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(b) Find the value of $\int_{1.6}^{3.4} f(x)\,dx$ from the following data using Simpson's $\frac{3}{8}th$ rule for the interval (1.6, 2.2) and $\frac{1}{3}rd$, rule for (2.2, 3.4) :

x	:	1.6	1.8	2.0	2.2	2.4
f(x)	:	4.953	6.050	7.389	9.025	11.023
x	:	2.6	2.8	3.0	3.2	3.4
f(x)	:	13.464	16.445	20.086	24.533	29.964

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(c) For the differential equation

$$\frac{dy}{dx}=y-x^2, y(0)=1$$

starting values are given as

y(0.2) = 1.2186, y(0.4) = 1.4682 and y(0.6) = 1.7379.

Using Milne's predictor corrector method advance the solution to x = 0.8 and compare it with the analytical solution. (Carry four decimals).

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13. (a) A and B play a game of dice. A wins if he throws 11 with 3 dice before B throws 7 with 2 dice. B wins is he throws 7 before A throws 11. A starts the game and they throw alternately. What are the odds against A winning the game ultimately?

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(b) One bag contains three identical cards marked 1, 2, 3 and another contains two cards marked 1, 2. Two cards are randomly chosen one from each bag and the numbers observed. Denote the minimum of the two by X and the sum of the two by Y. Find the joint probability distribution of the random variable pair (X, Y). Find also the two marginal distributions and the conditional distribution of X when $Y \leq 4$.

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(c) If $X_1 = Y_1+Y_2$, $X_2 = Y_2+Y_3$ and $X_3=Y_3+Y_1$, where Y_1, Y_2, Y_3 are uncorrelated variates with mean zero and standard deviation unity, find the multiple correlation of X_1 on X_2 and X_3 and partial correlation of X_1 and X_2.

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14. (a) Tests of fidelity and selectivity of 190 radio receivers produced the results shown in the following table:

		Fidelity			
		Low	**Average**	**High**	**Total**
Selectivity	**Low**	6	12	32	50
	Average	33	61	18	112
	High	13	15	0	28
	Total	52	88	50	190

Apply the chi-square test at significance level of 0.01 and show that there is indeed a relationship between fidelity and selectivity.
The values of chi-square at 1% level for various degrees of freedom are given below:

No. of degrees of freedom	Chi-square value
3	11.345
4	13.277
5	13.086
6	16.812
7	18.475

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(b) For the Gamma distribution given by the density function

$$f(x) = \begin{cases} \dfrac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1}e^{-x/\beta} & \text{for } x > 0, \alpha > 0, \beta > 0 \\ 0 & \text{otherwise} \end{cases}$$

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find the moment coefficient of skewness.

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(c) State and prove Chebyshev's lemma for a discrete distribution and deduce from it the weak law of large numbers.

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15. (a) Solve the following linear programming problem:

Maximize: $z = x_1 + 2x_2 + 3x_3 - x_4$

subject to

$$x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$
$$x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0.$$

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(b) Solve the transportation problem below for minimizing the cost:

Costs		Store 1	2	3	4	5	6	Availability
Warehouse	1	9	12	9	6	9	10	5
	2	7	3	7	7	5	5	6
	3	6	5	9	11	3	11	2
	4	6	8	11	2	2	10	9
Requirement		4	4	6	2	4	2	22

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(c) There are five jobs each of which must go through two machines A and B in the order A, B. Processing times are given below:

Job :	1	2	3	4	5
Time for A (in hours) :	7	3	11	5	12
Time for A (in hours) :	4	8	9	10	6

Determine a sequence for the jobs that will minimise the elapsed time. Compute the total idle times for the machines in this period.

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16 (a) Using dynamic programming technique solve the problem below:

Minimize $z = x_1^2 + x_2^2 + x_3^2$

subject to $x_1+x_2+x_3 \geq 30$

$x_1\geq 0, x_2\geq 0, x_3\geq 0.$

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(b) If P_n represents the probability of finding n in the long run in a queuing system with Poisson arrivals having parameter λ and exponential service times with parameter μ, show that

$$\lambda P_{n-1} - (\lambda+\mu)P_n + \mu P_{n+1} = 0 \text{ for } n > 0$$

and $-\lambda P_0 + \mu P_1 = 0$

Solve these difference equations and obtain P_n in terms of $\rho = \frac{\lambda}{\mu}$.

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(c) For a machine the following data are available.

Year	Cost of Spares (Rs.)	Salary of Maintenance Staff (Rs.)	Loss due to break-downs (Rs.)	Resale value (Rs.)
0	-	-	-	20000
1	100	1600	500	14000
2	500	1600	700	12000
3	700	1600	500	10000
4	900	2000	1000	6000
5	1300	2400	1500	3000
6	1600	2400	1600	800

Determine the optimum policy for replacement of the above machine.

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