MATHEMATICS

Time Allowed: 3 hours | ***Maximum Marks: 300***

Assume suitable data if considered and indicate the same clearly.
Candidates should attempt any five questions.
All questions carry equal marks.

PAPER-I

1. (a) Let V be the vector space of polynomials over R. Find a basis and dimension of the subspace W of V spanned by the polynomials
$$v_1 = t^3 - 2t^2 + 4t + 1, \quad v_2 = 2t^3 - 3t^2 + 9t - 1,\ v_3 = t^3 + 6t - 5,\ v_4 = 2t^3 - 5t^2 + 7t + 5.$$
   (b) Verify that the transformation defined by
$$T(x_1, x_2) = (x_1 + x_2, x_1 - x_2, x_2)$$
is a linear transformation from R^2 into R^3. Find its range, null space and nullity.
   (c) Let V be the vector space of 2 x 2 matrices over R. Determine whether the matrices A, B, C $\in$ V are dependent where
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 3 & -1 \\ 2 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & -5 \\ -4 & 0 \end{bmatrix}$$

2. (a) Let a square matrix A of order n be such that each of its diagonal elements is μ and each of its off diagonal elements is 1. If $B = \lambda A$ is orthogonal, determine the values of λ and μ.
   (b) Show that $A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & 0 \\ 2 & 2 & 3 \end{bmatrix}$ is diagoalisable over R and find a matrix P such that $P^{-1}AP$ is diagonal. Hence determine A^{25}.
   (c) Let $A = [a_{ij}]$ be a square matrix of order n such that $[a_{ij}] \leq M\ \forall\ i, j = 1, 2, \ldots n$. Let λ be an eigen-value of A. Show that $|\lambda| \leq nM$.

3. (a) Define a positive definite matrix. Show that a positive definite matrix is always non-singular. Prove that its converse does not hold.
   (b) Find the characteristic roots and their corresponding vectors for the matrix
$$\begin{bmatrix} 6 & -2 & 2 \\ -2 & 3 & -1 \\ 2 & -1 & 3 \end{bmatrix}$$
   (c) Find an invertible matrix P which reduces
$Q(x,y,z) = 2xy + 2yz + 2zx$ to its canonical form.

4. (a) Suppose
$$f(x) = 17x^{12} - 124x^9 + 16x^3 - 129x^2 + x - 1.$$
Determine $\frac{d}{dx}\left(f^{-1}\right)$ at x = -1 if it exists.
   (b) Prove that the volume of the greatest parallelopiped that can be inscribed in the ellipsoid
$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1 \text{ is } \frac{8abc}{3\sqrt{3}}$$

(c) Show that the asymptotes of the curve

$$(x^2-y^2)(y^2-4x^2) + 6x^3 - 5x^2y - 3xy^2 + zy^3 - x^2 + 3xy - 1 = 0$$

cut the curve again in eight points which lie on a circle of radius 1.

5. (a) An area bounded by a quadrant of a circle of radius a and the tangents at its extremities revolves about one of the tangents. Find the volume so generated.

(b) Show how the change of order in the integral $\int_0^\infty \int_0^\infty e^{-xy} \sin x \, dx \, dy$ leads to the evaluation of $\int_0^\infty \frac{\sin x}{x} dx$. Hence evaluate it.

(c) Show that in $\sqrt{n}\sqrt{n+\frac{1}{2}} = \frac{\sqrt{\pi}}{2^{2n-1}}$ where n > 0 and $\sqrt{n}$ denotes gamma function.

6. (a) Let P be a point on an ellipse with its centre at the point C. Let CD and CP be two conjugate diameters If the normal at P cuts CD in F, show that CD.PF is a constant and the locus of F is $\frac{a^2}{x^2} + \frac{b^2}{y^2} = \left[\frac{a^2-b^2}{x^2+y^2}\right]^2$ where $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ equation of the given ellipse.

(b) A circle passing through the focus of conic section whose latus rectum is $2l$ meets the conic in four points whose distances from the focus are γ_1, γ_2, γ_3 and γ_4 Prove that

$$\frac{1}{\gamma_1} + \frac{1}{\gamma_2} + \frac{1}{\gamma_3} + \frac{1}{\gamma_4} = \frac{2}{l}.$$

(c) Determine the curvature of the circular helix $\vec{r}(t) = (a\cos t)\hat{i} + (a\sin t)\hat{j} + (bt)\hat{k}$ and an equation of the normal plane at the point $\left(0, a, \frac{\pi b}{2}\right)$.

7. (a) Find the reflection of the plane x + y + z - 1= 0 in plane 3x + 4z + 1 = 0

(b) Show that the point of intersection of three mutually perpendicular tangent planes to the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ lies on the sphere $x^2 + y^2 + z^2 = a^2 + b^2 + c^2$.

(c) Find the equation of the spheres which pass through the circle $x^2 + y^2 + z^2 - 4x - y + 3z + 12 = 0$, $2x + 3y - 7z = 10$ and touch the plane $x - 2y + 2z = 1$.

8. (a) Solve the initial value problem

$$\frac{dy}{dx} = \frac{x}{x^2y + y^3}, y(0) = 0$$

(b) Solve

$(x^2-y^2+3x-y)dx + (x^2-y^2+x-3y)dy = 0$.

(c) Assume that a spherical rain drop evaporates at a rate proportional to its surface area. If its radius originally is 3 mm, and one hour later has been reduced to 2 mm. find an expression for the radius of the rain drop at any time.

9. (a) Solve

$$\frac{d^4y}{dx^4} + 6\frac{d^3y}{dx^3} + 11\frac{d^2y}{dx^2} 6\frac{dy}{dx} = 20e^{-2x}\sin x$$

(b) Make use of the transformation y(x) = u (x) sec x to obtain the solution of

$$y'' - 2y' \tan x + 5y = 0, y'(0) = 0,. y'(0) = \sqrt{6}$$

(c) Solve

$$(1+2x)^2 \frac{d^2y}{dx^2} - 6(1+2x)\frac{dy}{dx} + 16y = 8(1+2x)^2$$

$$y(0) = 0, y'(0) = 2.$$

10. (a) Prove that if $\vec{A}, \vec{B}$ and $\vec{C}$ are three given non-coplanar vectors, then any vector $\vec{F}$ can be put in the form $F = \alpha \vec{B} \times \vec{C} + \beta \vec{C} \times \vec{A} + \gamma \vec{A} \times \vec{B}$. For given determine α,β,γ.

(b) Verify Gauss theorem for $\vec{F} = 4x\hat{i} - 2y^2\hat{j} + z^2\hat{k}$ taken over the region bounded by $x^2 + y^2 = 4$, $z = 0$ and $z = 3$.

(c) Prove that the decomposition of a tensor into a symmetric and an anti-symmetric part is unique. Further show that the contracted product, S_{ij} T_{ij} of a tensor T_{ij} with a symmetric tensor S_{ij} is independent of the anti-symmetric part of T_{ij}.

11. (a) A heavy uniform chain rests on a rough cycloid whose axis vertical and vertex upwards, one end of the chain being at the vertex and the other at a cusp. If the equilibrium is limiting, show that $(1 + \mu^2)\, e^{\mu x/2} = 3$.

(b) A solid frustum of a paraboloid of revolution of height hand latus rectum 4a, rests with its vertex on the vertex of another parboloid (inverted) of revolution whose latus rectum is 4b. Show that the equilibrium is state if $h < \frac{3ab}{a+b}$.

(c) A cylinder of wood (specific gravity $\frac{3}{4}$) of height h, floats with its axis vertical in water and oil (specific gravity $\frac{1}{2}$). The length of the solid in contact with the oil is a $\left(< \frac{h}{2} \right)$. Find how much of the wood is above the liquid. Also find to what additional depth much oil be added so to cover the cylinder.

12. (a) A shell bursts on contact with the ground and pieces from it fly in all directions with all velocities upto 80 units. Show that a man 100 units away is in danger for a time of $\frac{5}{2}\sqrt{2}$ units if g is assumed to be of 32 units.

(b) A particle moves under a force $m\mu\,\{3au^4 — 2(a^2— b^2)u^5\}$, a > b and is projected from an apse at a distance a+b with velocity $\sqrt{\frac{\mu}{a+b}}$. Find the orbit.

(c) A particle is projected along the inner side of a smooth circle of radius a, the velocity at the lowest point being u. Show that if $2ag < u^2 < 5ag$, the particle will leave the circle before arriving at the highest point. What is the nature of the path after the particle leaves the circle?

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PAPER - I

SECTION - A

1. (a) Show that a necessary and sufficient condition for a subset H of a group G to be a subgroup is $HH^{-1} = H$.

 (b) Show that the order of each subgroup of a finite group is a divisor of the order of the group.

 (c) In a group G, the commutator (a., b),a, b $\in$ G is the element $aba^{-1}b^{-1}$ and the smallest subgroup containing all commutators is called the commutator subgroup of G. Show that a quotient group G/H is abelian if and only if H contains the commutator subgroup of G.

2. (a) If x^2=x for all x in a ring R, show that R is commutative. Give an example to show that the converse is not true.

 (b) Show that an ideal S of the ring of integers Z is a maximal ideal if and only if S is generated by a prime integer.

 (c) Show that in an integral domain every prime element is irreducible. Give an example to show that the converse is not true.

3. (a) Show that the set C of all complex numbers is a metric space with respect to the metric d, defined by

$$d(z_1, z_2) = \frac{|z_1 - z_2|}{\sqrt{(1+|z_1|^2)(1+|z_2|^2)}}$$

 for all $z_1, z_2 \in C$.

 (b) f is a mapping of a metric space X into another metric space Y. Show that f is continuous if and only if

$$x_n \to x_0 = f(x_n) \to f(x_0)$$

 (c) Show that a non-empty set Pin R^n each of whose points is a limit-point is uncountable.

4. (a) Show that $\iiint_D xyz\,dx\,dy\,dz = \frac{a^2b^2c^2}{6}$ where domain D is given by $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{Z^2}{c^2} \le 1$

 (b) If $u = \sin^{-1}[(x^2 + y^2)^{1/5}]$, prove that

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y.\partial x} + y^2\frac{\partial^2 x}{\partial y^2} = \frac{2}{25}\tan u\left(2\tan^2 u - 3\right)$$

(c) Suppose f is Riemann-Stieltjes integrable with respect to α on [a, b], $m \le f \le M$, ϕ is continuous on [m, M] and $h(x) = \phi[f(x)]$ on [a, b]. Prove that h is Riemann-Stieltjes integrable with respect to α on [a, b].

5. (a) Prove that $u = e^x (x \cos y - y \sin y)$ is harmonic and find the analytic function whose real part is u.

(b) Evaluate $\oint_C \frac{dz}{z+2}$ where C is unit circle.

Deduce that $\int_0^{2\pi} \frac{1+2\cos\theta}{5+4\cos\theta} d\theta = 0$.

(c) If $f(z) = \frac{A_1}{z-a} + \frac{A_2}{(z-a)^2} + ... + \frac{A_n}{(z-a)^n}$ find the residue at a for $\frac{f(z)}{z-b}$ where $A_1, A_2, \ldots A_n$, a and b are constants. What is the residue at infinity?

6. (a) Find the Laurent series for the function $e^{1/z}$ in $0 < |z| < \infty$. Deduce that

$$\frac{1}{\pi}\int_0^{\pi} \exp(\cos\theta).\cos(\sin\theta - n\theta)\, d\theta = \frac{1}{n!}, \qquad (n = 0,1,2,......)$$

(b) Integrating e^{-z^2} along a suitable rectangular contour show that

$$\int_0^{\infty} e^{-x^2} \cos 2bx\, dx = \frac{\sqrt{\pi}}{2} e^{-b^2}$$

(c) Find the function f(z) analytic with in the unit circle, which takes the values $\frac{a - \cos\theta + i\sin\theta}{a^2 - 2a\cos\theta + 1}\, 0 \le \theta \le 2\pi$ on the circle.

7. (a) (i) Find the differential equation of all surfaces of revolution having z-axis as the axis of rotation.

(ii) Form the differential equation by eliminating a and b from $z = (x^2 + a)(y^2 + b)$

(b) Find the equation of surfaces satisfying $4yzp + q + 2y = 0$ and passing through $y^2 + z^2 = 1$, $x + z = 2$

(c) Solve $(y+z)p + (z+x)q = x+y$

8. (a) Use Charpit's method to find complete integral of $z^2(p^2 z^2 + q^2) = 1$

(b) Solve : $(D_X^3 - D_Y^3)z = x^3y^3$

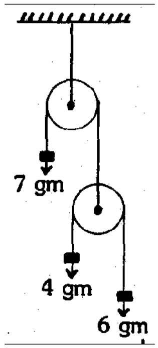
(c) Apply Jacobi's method to find complete integral of $p_1^3+p_2^2+p_3=1$. Here $p_1 = \frac{\partial z}{\partial x_1}, p_2 = \frac{\partial z}{\partial x_2}, p_3 = \frac{\partial z}{\partial x_3}$ and z is a function of x_1, x_2, x_3.

SECTION - B

9. (a) Using Lagrange's equations obtain the differential equations for planetary motion.

(b) A circular, cylinder of radius 3 cm and weight W whose center of gravity, G, is at a distance 1 cm from the axis, rolls on a horizontal plane. If motion be just started from the position of unstable equilibrium, show that the normal reaction of the plane when G, is in its lowest position is $W(2 + k^2)/(1 + k^2)$, where 2k is the radius of gyration about an axis passing through G.

(c) A pulley system is given as shown , in the diagram. Discuss the motion of the system using Lagrange's method when the pulley wheels have negligible masses and moments of inertia and their wheels are frictionless.



10. (a) A solid homogeneous sphere is rolling on the inside of a fixed hollow sphere, the two centres being always in the same vertical plane. Show that the smaller sphere will make complete revolution if, when it is in its lowest position, the pressure on it is greater then $\frac{34}{7}$ times its own weight.

(b) Three equal uniform rods AB, BC, CD are smoothly joined at B and C and the ends A and D are fastened to smooth fixed points whose distance apart is equal to the length of the rod. The frame being at rest is in the form of the square. A blow J is given perpendicular to AB at its middle point and in the plane of the square. Show that the energy set up is $\frac{3J^2}{40m}$ where m is the mass of each rod. Find also the blows at the joints B and C.

11. (a) Show that $\frac{x^2}{a^2}\tan^2 t + \frac{y^2}{b^2}\cot^2 t = 1$

is a possible form for the bounding surface of a liquid, and find an expression for the normal velocity.

(b) A stream in a horizontal pipe, after passing a contraction in the pipe, at which its cross-sectional area is A, is delivered at the atmospheric pressure at a place where the cross-sectional area is B. Show that if a side tube is connected with the pipe at the former place, water will be sucked up through it into pipe from a reservoir at a depth $\frac{s^2}{2g}\left(\frac{1}{A^2} - \frac{1}{B^2}\right)$ below the pipe where s is the delivery per second.

(c) Using the method of images prove that if there be a source m at the point z_0 in a fluid bounded by the lines θ = 0 and $\theta = \frac{\pi}{3}$, the solution is usual notations, $\phi + i\psi = -m\log\left[\left(z^3 - z_0^3\right)\left(z^3 - z_0'^3\right)\right]$ where $z_0 = x_0 + i\,y_0$ and $z_0 = x_0 - i\,y_0$.

12. (a) Apply that fourth order Runge-Kutta method to find a value of y correct to four places of decimals at x = 0.2, when

$$y' = \frac{dy}{dx} = x + y,\ y(0) = 1$$

(b) Show that the iteration formula for finding the reciprocal of N is $x_{n+1} = x_n\,(2 - N_{xn})$, n = 0, 1…

(c) Obtain the cubic spline approximation for the function given in the tabular form below:

x :	0	1	2	3
f(x) :	1	2	33	244

and $M_0 = 0$, $M_3 = 0$.

13. (a) Let the variable X have the distribution $P(X = 0) = P(X=2)=p$, $P(X=1)=1-2p$, for $0 \le p \le \frac{1}{2}$. For what value of p is the variance of X a minimum?

(b) If A and B are any two events and the probability P(B)#1, prove that $P(A/\overline{B}) = \frac{P(A)-P(A\cap B)}{1-P(B)}$ and hence deduce that $P(A\cap B) \ge P(A) + P(B) - 1$.

(c) The random variables $Y_1, Y_2, ..., Y_n$, have equal expectations and finite variances. Is the weak law of large numbers applicable to this sequence if all the covariances are negative?

14. (a) If X is a Poisson variable with parameter λ and χ^2 is a chi-square variate with 2k degrees of freedom, prove that for a positive integer k, $P[X \le k - 1) = P[\chi^2 > 2\lambda]$.

(b) Let X and Y be two random variables taking three values -1, 0, 1, and having the joint probability distribution given below:

X→ / ↓Y	-1	0	1	**Total**
-1	0	0.1	0.1	0.2
0	0.2	0.2	0.2	0.6
1	0	0.1	0.1	0.2
Total	0.2	0.4	0.4	1

(i) Find E(X), E(Y) and E(XY). Are X and Y independent?

(ii) Give that Y = 0, what is the conditional probability distribution of X?

(c) If X and Y are independent normal variates with means m_1, m_2 and variances σ_1^2, σ_2^2 respectively, determine the probability distribution $Z = \frac{(X-m_1)}{(Y-m_2)}$ name the distribution obtained.

15. (a) Describe a quadratic programming problem and outline a method of solving it.

(b) Minimize $u_1^2 + u_2^2 + u_3^2$

Subject to $u_1 + u_2 + u_3 \ge 10$; $u_1, u_2, u_3, \ge 0$.

(c) Consider a modified form of matching biased coins game problem. The matching player is paid eight rupees if the two coins turn both heads and one rupees if the coins turn both tails. The non-matching player as paid three rupees when the two coins do not match. Given the choice of being the matching or non- matching players, which one would you choose and what would be your strategy?

16. (a) State the Transportation problem in general terms and explain the problem of degeneracy.

(b) Use simplex method to solve the following Linear Programming Problem:

Maximize $z = 4x_1 + 10x_2$

subject to the constraints $2x_1 + x_2 \leq 50$

$2x_1 + 5x_2 \leq 100$

$2x_1 + 3x_2 \leq 90$

$x_1, x_2 \geq 0.$

(c) In a factory, there are six jobs to perform and each should go through two machines A and B in the order A, B. The processing timings (in hours) for the jobs are given below. Determine the sequence for performing the jobs that would minimize the total elapsed time T. What is the value of T?

Job :	J_1	J_2	J_3	J_4	J_5	J_6
Machine A :	1	3	8	5	6	3
Machine B :	5	6	3	2	2	10

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